Hill's Elements of Rhetoric and Composition........

By D. J. Hill, A.M., President Lewisburg University, author of the Science of Rhetoric. Beginning with the selection of a theme, this book conducts the learner through every process of composition, including the accumulation of material, its arrangement, the choice of words, the construction of sentences, the variati formation of paragraph .e criticism of the compl

Hill's Sci ....

An intr iscourse. By D. J. Hill Lewisburg. 12mo, 300

This is a for advanced classes.

Intellectu pages......

By Fra own University.

The Elem ....

By Fra own University, and th thousand. 12mo, clot

Elements ....

By Fra Brown University. 12mo, cloth, 403 pages.

Recast by Aaron L. Chapin, D.D., President of Beloit College.

No text-book on the subject has gained such general acceptance, and been so extensively and continuously used, as Dr. Wayland's. Dr. Chapin has had chiefly in mind the *wants of the class-room,* as suggested by an experience of many years. His aim has been to give in full and proportioned, yet clear and compact statement, the elements of this important branch of science, in their latest aspects and applications.

G. A. Wentworth

ELEMENTS

OF

GEOMETRY,

AND THE

FIRST PRINCIPLES

OF MODERN GEOMETRY.

BY

WM. H. H. PHILLIPS, PH. D.,

TEACHER OF MATHEMATICS IN WESLEYAN ACADEMY.

SECOND EDITION.

NEW YORK:
SHELDON & COMPANY,
8 MURRAY STREET.
1878.

Miss Ellen L. Wentworth

THE AUTHOR

RESPECTFULLY DEDICATES THIS WORK

TO HIS FRIEND,

CAPTAIN OLIVER CUTTS,

WHOSE INFLUENCE AND MUNIFICENCE HAVE GREATLY AIDED

THE CAUSE OF HIGHER EDUCATION.

PREFACE.

The following are some of the distinguishing features of this work:—

A line is considered the path of a moving point; and a surface is supposed to be generated by a moving line. These lead to the introduction of the term "locus," which is frequently used to shorten and generalize a demonstration. An angle is supposed to be generated by the revolution of a side about the vertex; and is defined as the difference of direction of two lines drawn from the same point. It is taken in this sense in surveying, astronomy, and all applied mathematics. As to the propriety of introducing the idea of motion in these fundamental definitions, I find but one sentiment among those who sympathize with the modern methods of geometry. Indeed, the demand for it induced me, more than any thing else, to commence the preparation of a new text-book.

The theories of Limits, Harmonic Proportion, Transversals, and Polars, and the Principle of Continuity, which play so important a part in modern geometry, are briefly stated and applied in connection with the propositions from which they naturally follow. They are so interwoven into higher mathematics, that it has become necessary for the student's progress to introduce them into our text-books of elementary geometry. Book IX. is a brief introduction to modern geometry.

The articles marked with a star may be omitted, without destroying the connection; but I have endeavored to give them in so simple and attractive a manner, that few will desire to do so.

A complete classification has been made; and each subject is treated by itself as distinctly as possible. It has been my object to make the

definitions and demonstrations short and concise, without sacrificing perspicuity. My own views in regard to "condensing the demonstration by means of symbols" were confirmed by Prof. J. R. French of Syracuse University, who kindly examined the first pages of my manuscript, and the plan of the work. A student will often give a demonstration correctly, without being able to state afterwards the hypothesis, or what he has proved. These are therefore given in a distinct form. Exercises and numerical examples are inserted in numbers sufficiently great to fix the truth of the proposition in the mind of the student, without materially increasing the amount of work to be done. In solid geometry I have shown the relation of spherical polygons to polyhedral angles whose vertices are at the centre of the sphere; so that, when any proposition is proved in reference to the one, we may immediately infer a corresponding proposition in reference to the other, without a separate demonstration.

My constant aim has been, first, to give the leading features of the latest and most approved text-books of this country and Europe; second, to put every proposition in the simplest form possible, and at the same time to make suggestions that will lead the student to investigate for himself. More discipline is acquired by working out a single demonstration without aid than by learning several propositions that are fully given in the text-book. Most students depend too much upon books, and too little upon their own mental powers, — an error which ought to be avoided, especially in the study of geometry. This work is not designed for students who learn merely by rote. Every reference must be quoted, and its application shown.

Besides several foreign works, that by Prof. Chauvenet has been my constant companion.

To my friend and former teacher, Prof. J. M. Van Vleck of Wesleyan University, I am especially indebted for the invaluable aid he has kindly rendered me during the preparation of this work.

WM. H. H. PHILLIPS.

WILBRAHAM, MASS., July 1, 1874.

CONTENTS.

PLANE GEOMETRY.

SOLID GEOMETRY.

ELEMENTS OF GEOMETRY.

GENERAL PRINCIPLES.

DEFINITIONS.

1. **Space** is unlimited extension in all directions. A limited portion of space is called a **geometrical solid.** A **physical solid** is the material occupying the space. Geometry treats only of the form and magnitude of the solid. The term **solid,** in this work, will signify a geometrical solid.

2. A solid has extension in all directions from any point within; but it is sufficient to consider three dimensions,—*length*, *breadth*, and *thickness.*

3. The limits of a solid are called **surfaces.** They have only two dimensions, *length* and *breadth;* and are, therefore, no part of the solid.

4. The limits of a surface are called **lines.** They have only one dimension, *length;* and are no part of the surface which they bound.

5. The entire limit of a surface is called its **perimeter.**

6. The limits of a line are called **points.** They are no part of the line, and have neither length, breadth, nor thickness.

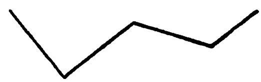
7. We may consider a point as merely a position in space; a line, as the path of a moving point; a surface, as generated by a moving line; and a solid, as generated by a moving surface.

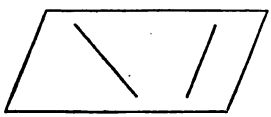
8. The direction in which the point is supposed to have moved in generating a line may be considered positive; the opposite direction, negative. This may be indicated by the order of the letters; as, AB = — BA.

9. A **straight line** is one that does not change its direction at any point. It is sometimes called simply a **line**.

10. A **curved line** is one that changes its direction at every point.

11. A **broken line** is a line composed of straight lines lying in different directions.





12. A surface is called a **plane** when the straight line joining any two of its points lies wholly in the surface.

13. A **curved surface** is a surface which is neither a plane, nor composed of planes; as the surface of a ball, pipe, &c.

14. A **geometrical figure** is a combination of points, lines, surfaces, or solids, formed under certain conditions. A **plane figure** is a combination of points and lines which are confined to one and the same plane.

15. A **rectilinear figure** is one formed by straight lines.

16. **Geometry** is that part of mathematics which treats of the properties, measurement, and construction of figures.

17. **Plane geometry** treats of plane figures.

18. **Solid geometry** treats of figures which represent points, lines, surfaces, and solids that are not confined to the same plane.

TERMS.

1. An **axiom** is a self-evident truth.
2. A **theorem** is a truth requiring demonstration.
3. A **problem** is a question requiring a solution.

4. A **postulate** is a problem the possibility of whose solution is self-evident.

5. A **lemma** is an auxiliary theorem.

6. Axioms, theorems, problems, and postulates are called **propositions.**

7. A **corollary** is an obvious consequence of one or more propositions or other premises.

8. A **scholium** is a remark made upon one or more propositions, showing their connection, use, extension, or limitation.

9. An **hypothesis** is a supposition made either in the statement of a proposition or in the course of a demonstration.

AXIOMS.

1. Things which are equal to the same thing are equal to each other.

2. If equals be added to equals, the sums will be equal.

3. If equals be subtracted from equals, the remainders will be equal.

4. If equals be added to unequals, the sums will be unequal.

5. If equals be subtracted from unequals, the remainders will be unequal.

6. If equals be multiplied by equals, the products will be equal.

7. If equals be divided by equals, the quotients will be equal.

8. The whole is greater than any of its parts.

9. The whole is equal to the sum of all its parts.

Illustrate these axioms with the equations $a = b$, $c = d$, &c.

TABLE OF SYMBOLS USED IN THIS WORK.

$+$ is the sign of addition; thus, $a+b$ indicates that b is to be added to a, and is read a plus b.

$-$ is the sign of subtraction; thus, $a-b$ indicates that b is to be subtracted from a, and is read a minus b.

$\times$ is the sign of multiplication.

$\div$ is the sign of division; also $\frac{a}{b}$ indicates that a is to be divided by b.

a^2, a^3, . . . indicates that a is to be raised to the second, third, . . . power.

$\sqrt{a}$, $\sqrt[3]{a}$, . . . indicates that the square root, cube root, . . . of a is to be taken.

$=$ is the sign of equality; thus, $a=b$ is read a is equal to b.

$>$ or $<$ is the sign of inequality; thus, $a>b$ is read, a is greater than b, and $a<b$ is read, a is less than b.

$\angle$ is used for the word *angle*, p. 6.

R is used for the word *right-angle*, p. 6.

$\triangle$ is used for the word *triangle*, p. 15.

$\perp$ designates perpendicularity, p. 6.

$\parallel$ designates parallelism, p. 9.

$\cong$ designates congruity, p. 14.

$\sim$ designates similarity, p. 83.

In the references, Roman numerals refer to books, and Arabic numerals to articles; thus (I., 38) refers to Book I., Article XXXVIII. (18) refers to Article XVIII. of the same book.

PLANE GEOMETRY.

BOOK I.

LINES AND ANGLES.

I.—POSTULATES.

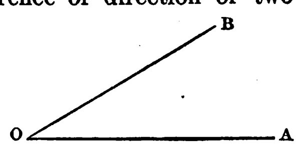
1. A straight line may be drawn between any two points.
2. A straight line may be prolonged to any length.
3. From the longer of two straight lines a part may be cut off equal to the less.
4. A straight line may be bisected; that is, divided into two equal parts.

II.—AXIOMS.

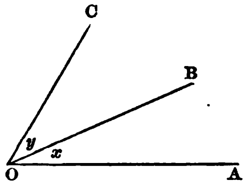
1. A straight line is the shortest distance between two points.
2. But one straight line can be drawn between two points.
3. Two straight lines that have two points common coincide throughout their whole extent, and form one and the same straight line.
4. Two straight lines can cut each other in but one point.

III.

DEF. 1. An **angle** is the difference of direction of two straight lines drawn from the same point; as, AOB. The common point, O, is called the **vertex**; the lines OA and OB, the **sides** of the angle.



An angle may be designated by the letter at the vertex when no other angle has the same vertex; as, the angle O:

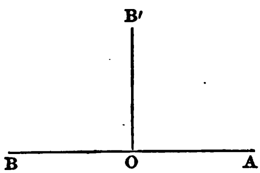


but, when two or more angles have the same vertex, they must be designated by letters in the opening, or by three letters, the two on the sides, and the one at the vertex between them; as, angle x or AOB, angle y or BOC.

Angles may be added; as, AOB + BOC = AOC.

Def. 2. The sign $\angle$ stands for angle; as, $\angle$AOB. If the line OB be turned about the point O towards OA, the angle AOB will gradually diminish and become zero when OB corresponds with, or has the same direction as, OA. If OB be turned in the opposite direction about the point O, the angle AOB will increase. When OB lies in the opposite direction of OA, the angle AOB is called an **extended angle.**

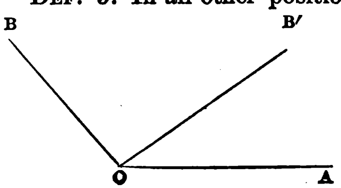
The two sides of an extended angle lie in the same straight line.



Def. 3. A **right angle** is half an extended angle. It is designated simply by the letter R; as, $\angle$AOB′ = B′OB = R.

Def. 4. When one straight line meets another so as to form a right angle, the lines are said to be **perpendicular** to each other; as OB′ is perpendicular to OA, which may be expressed by the sign OB′ $\perp$ OA; also OA $\perp$ OB′.

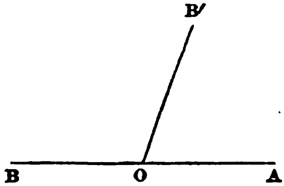
Def. 5. In all other positions, OB is called an **oblique line**; and the angles it forms with OA, **oblique angles**; as, $\angle$AOB′, or $\angle$AOB.



Def. 6. An angle less than a right angle is called an **acute angle**; as, AOB′.

Def. 7. An angle greater than a right angle is called an **obtuse angle**; as, AOB.

Def. 8. Two angles which have the same vertex, and a common side between them, are called **contiguous angles**; as, $\angle$ AOB′ and $\angle$ B′OB. When the two outer sides, OA and OB, lie in the same straight line, they are called **adjacent angles.**



Cor. 1. All extended angles are equal; for they may be so laid upon each other that their sides will correspond.

Cor. 2. All right angles are equal (3, 3).

Cor. 3. From a point in a straight line, only one perpendicular can be drawn to that line; for, in the revolution of the line OB′ about the point O, there can be but one position in which $\angle$ AOB′ $= \frac{1}{2}$ AOB $=$ R. By the term angle, we will understand an angle less than an extended angle.

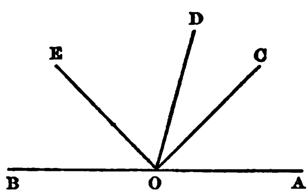
IV.

Theorem. *Two adjacent angles are together equal to two right angles.*

For the two adjacent angles, AOB′ and B′OB, are together equal to the extended angle AOB $=$ 2R for every position of the line OB′.

Cor. 1. If one of two adjacent angles is a right angle, the other is also a right angle. If two adjacent angles are equal, each is a right angle.

Cor. 2. The sum of all the angles at a point on the same side of a straight line are together equal to two right angles; for AOC $+$ COD $+$ DOE $+$ EOB $=$ AOB $=$ 2R.

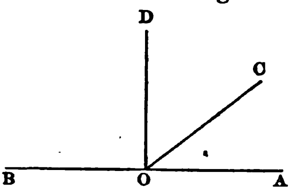


Cor. 3. The sum of all the angles that may be formed at

the point O, upon the other side of AB, is also equal to two right angles. Hence *all the angles formed about a point are together equal to four right angles.*

V.

Def. 1. Two angles are said to be *complements* of each other when their sum is equal to one right angle; as, ∠AOC and ∠COD.

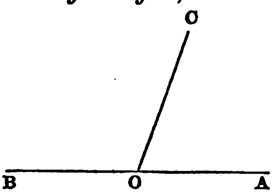


Def. 2. Two angles are said to be *supplements* of each other when their sum is equal to two right angles; as, ∠AOC and COB.

Cor. 3. It is clear that the complements of equal angles are equal to each other, and that the supplements of equal angles are equal to each other.

VI.

Theorem. *If the sum of two contiguous angles is equal to two right angles, the two outer sides form one straight line.*



Hypothesis. $AOC + COB = 2R$.

To Prove. BO and OA form one straight line.

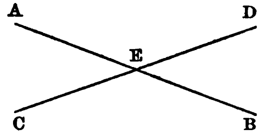
Proof. The angle AOC is the supplement of COB (5, 2).* But the angle which CO makes with the prolongation of BO is also the supplement of COB (4), and is therefore equal to the angle AOC (5, 3).

Hence BO and OA form one straight line.

* Let the student quote every reference in full, and show its application.

VII.

Theorem. *If two straight lines cut each other, the opposite or vertical angles are equal.*



HYPOTH. The line AB cuts CD.

TO PROVE. $\angle$ AEC $=$ DEB.

PROOF. $\angle$ AEC $+$ AED $=$ 2R (4);

also $\angle$ DEB $+$ AED $=$ 2R (4):

hence $\angle$ AEC $+$ AED $=$ DEB $+$ AED (Ax. 1);

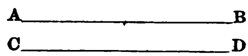
and $\angle$ AEC $=$ DEB (Ax. 3).

COR. If one of the angles is a right angle, the three remaining angles will be right angles also.

PARALLEL LINES.

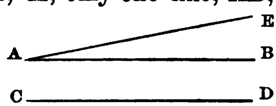
VIII.

DEF. 1. Straight lines which have the same direction are called **parallel lines**. The sign $\parallel$ stands for parallel; as, AB $\parallel$ CD.



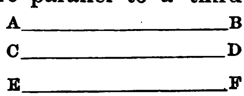
AXIOM. Two parallel lines lie in the same plane, and cannot meet, however far both be produced.

COR. 1. Through the same point, A, only one line, AB, can be drawn parallel to a given line, CD; for any other line, AE, must have a different direction from AB, and, consequently, from CD also.



COR. 2. If a straight line cuts one of two parallel lines, it will cut the other also, if sufficiently produced; for, if not, it would have the same direction, and be parallel to it.

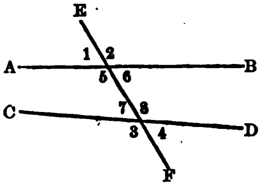
COR. 3. Two straight lines which are parallel to a third line are parallel to each other. If AB and CD $\parallel$ EF, then AB $\parallel$ CD; for the three straight lines have the same direction.



Def. 2. Two straight lines that are not parallel are said to *converge* in the direction they approach each other, and *diverge* in the opposite direction.

IX.

Def. 1. When two straight lines, AB and CD, are cut by a third line, EF, eight angles are formed, — four **exterior** angles, 1, 2, 3, 4; and four **interior** angles, 5, 6, 7, 8. They are divided into the following pairs: —



Def. 2. **Corresponding angles** are an exterior and interior angle on the same side of the cutting line, but not adjacent; as, 2 and 8, 6 and 4, 1 and 7, 5 and 3.

Def. 3. **Alternate angles** are two exterior angles or two interior angles on opposite sides of the secant line, but not adjacent; as, 1 and 4, 2 and 3, 5 and 8, 6 and 7.

Def. 4. **Opposite angles** are two exterior or two interior angles upon the same side of the cutting line; as, 1 and 3, 2 and 4, 5 and 7, 6 and 8.

Def. 5. The cutting line EF is called a **secant.**

X.

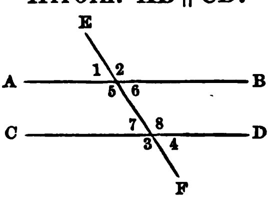
Theorem. *If two parallels are cut by a third line,*

1. *The corresponding angles are equal.*

2. *The alternate angles are equal.*

3. *The sum of two opposite angles is equal to two right angles.*

Hypoth. AB || CD.



To be Proved. 1. $\angle 2 = 8$, $\angle 6 = 4 \ldots.$

Proof. As AB and CD have the same direction (8), their difference of direction with a third line, EF, will be the same; that is,

$$\angle 2 = 8, \angle 6 = 4 \ldots.$$

TO BE PROVED. 2. $\angle 5 = 8$, $\angle 6 = 7$, $\angle 2 = 3$, $\angle 1 = 4$.

PROOF. $\angle 2 = 8$ (Case 1),

$\angle 2 = 5$ (7) :*

hence $\angle 5 = 8$ (Ax. 1).

In a similar manner, we may prove the other pairs of alternate angles equal.

TO BE PROVED. 3. $\angle 6 + 8 = 2R$ . . . .

PROOF. $\angle 2 = 8$ (Case 1),

$\angle 6 + 2 = 2R$ (4) :

hence, by substitution, $\angle 6 + 8 = 2R$.

COR. If $\angle 6$ is a right angle, $\angle 8$ will be a right angle: hence, *if a line is perpendicular to one of two parallels, it is perpendicular to the other also.*

SCHOLIUM. When the secant cuts the parallels obliquely, there are formed four equal acute angles and four equal obtuse angles.

EXERCISE. Prove $\angle 5 + 7 = 2R$, and $\angle 2 + 4 = 2R$.

XI.

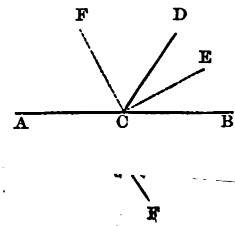
Theorem. Conversely, *two straight lines are parallel when a third line cutting them makes,* —

1. *The corresponding angles equal.*
2. *The alternate angles equal.*
3. *The sum of two opposite angles*

HYPOTH. 1. $\angle 2 = 8$.

TO BE PROVED. AB ‖ CD.

PROOF. Since $\angle 2 = 8$, the difference of direction of AB and FE is the same as that of CD and FE hence AB and CD have the same direction, and AB ‖ CD.



HYPOTH. 2. $\angle 5 = 8$.

* The student should be prepared to demonstrate any proposition referred to.

TO BE PROVED. AB ∥ CD.

PROOF. $\angle 5 = 8$ (Hypoth.),

$\angle 5 = 2$ (7):

hence $\angle 2 = 8$ (Ax. 1),

and AB ∥ CD (Case 1).

HYPOTH. 3. $\angle 6 + 8 = 2R$.

TO BE PROVED. AB ∥ CD.

PROOF. $\angle 6 + 8 = 2R$ (Hypoth.),

$\angle 6 + 2 = 2R$ (4):

hence $\angle 2 = 8$,

and AB ∥ CD (Case 1).

COR. 1. If AB and CD are each perpendicular to EF, $\angle 2 = 8$, being right angles: hence *two lines that are perpendicular to a third line are parallel to each other.*

COR. 2. When $\angle 6 + 8 < 2R$, it is evident that the lines AB and CD converge to the right: hence two lines converge on that side of the secant line on which the sum of the two opposite interior angles is less than two right angles.

XII.

Theorem. *If two angles have their sides respectively parallel, and lying both in the same or both in opposite directions, they are equal.*

Theorem. *If two pai*

1. *The corresponding*
2. *The alternate angle*
3. *The sum of two o*

angles.

HYPOTH. AB ∥ CD.

HYPOTH. BA ∥ ED, and BC ∥ EF.

TO BE PROVED. $\angle ABC = DEF$.

PROOF. Prolong ED, if necessary, until it cuts BC.

Then $\angle ABC = GDC$ (10, 1),

and $\angle DEF = GDC$ (10, 1):

hence $\angle ABC = DEF$ (Ax. 1).

If both the sides lie in opposite directions, as in D′EF′ and ABC,

then $\angle D'EF' = DEF$ (7):

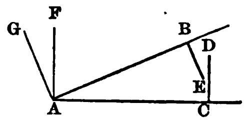
hence $\angle D'EF' = ABC$.

COR. If one pair of parallel sides lie in the same direction, and the other pair in the opposite direction, as in DEF′ and ABC, the angles are supplements of each other.

XIII.

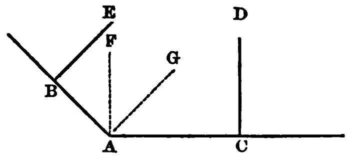
Theorem. *Two perpendiculars erected on the sides of an angle (not an extended angle) will meet, if sufficiently produced.*

HYPOTH. ∠BAC is the given angle; DC ⊥ AC, and EB ⊥ AB.



TO BE PROVED. DC and EB will meet, if produced.

PROOF. Through the vertex A, draw AF ∥ CD, and AG ∥ BE.



Then ∠FAC = R, since AF ∥ CD, and CD ⊥ AC:

hence ∠FAB <or> R;

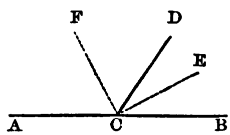
but ∠GAB = R, since AG ∥ BE, and BE ⊥ AB:

hence ∠GAB >or< FAB; and the two lines, AG and AF, have not the same direction.

Consequently, BE and CD, which are respectively parallel to them, have not the same direction, and will meet if sufficiently produced.

EXERCISE 1. Prove this by an auxiliary line, BC.

EXERCISE 2. The lines bisecting two adjacent angles are perpendicular to each other.



HYPOTH. EC and FC bisect the adjacent angles BCD and ACD.

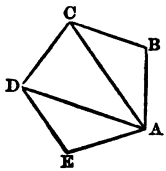
TO BE PROVED. ∠ECF = R.

EXERCISE 3. Prolong BE and CD until they cut in the point O. Prove that ∠BOC + BAC = 2 R.

POLYGONS.

XIV.

Def. 1. A **polygon** is a portion of a plane enclosed by straight lines. The lines are called *sides*, and their sum the *perimeter* of the polygon.



Def. 2. A *diagonal* is a line joining the vertices of two angles not consecutive; as, AC.

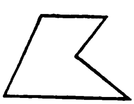
Def. 3. A polygon of three sides is called a *triangle;* one of four sides, a *quadrilateral;* one of five sides, a *pentagon;* one of six sides, a *hexagon;* one of seven sides, a *heptagon;* one of eight sides, an *octagon;* one of nine sides, an *enneagon;* one of ten sides, a *decagon;* one of twelve sides, a *dodecagon;* &c.

Def. 4. An *equilateral* polygon is one, all of whose sides are equal; and an *equiangular* polygon is one, all of whose angles are equal.

Def. 5. Two polygons are *mutually equilateral*, or *mutually equiangular*, when their sides or angles respectively taken in the same order are equal each to each.

Def. 6. Two polygons are *congruent* when they are both mutually equilateral and equiangular. It is evident that they may be applied to each other so as to correspond in all their parts. Congruity is expressed by the sign $\cong$.

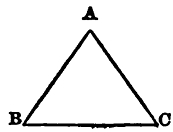
Def. 7. A **convex** polygon is one, each of whose angles is less than an extended angle.



Def. 8. A **concave** polygon is a polygon one or more of whose angles is greater than an extended angle. It is evident that a line may be drawn cutting the perimeter of a concave polygon in four or more points.

TRIANGLES.

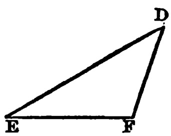
XV.



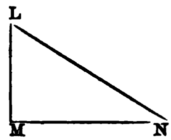
DEF. 1. An **isosceles** triangle is one which has two equal sides; as, ABC.

DEF. 2. A **scalene** triangle is one which has no two equal sides; as, DEF.

DEF. 3. An **acute-angled** triangle has three acute angles; as, ABC.



DEF. 4. An **obtuse-angled** triangle has an obtuse angle; as, DFE.



DEF. 5. A **right-angled** triangle has a right angle; as, LMN. The side opposite the right angle is called the *hypothenuse;* as, LN.

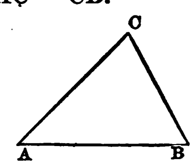
DEF. 6. The *base* of a triangle is the side upon which it is supposed to stand, and the opposite angle is called the *vertical angle.* Any side may be taken as the base. The sign $\triangle$ is often used for the word triangle.

XVI.

Theorem. *Any side of a triangle is less than the sum of the other two sides, and greater than their difference.*

HYPOTH. Let AB be any side of the triangle, ABC.

TO BE PROVED. $AB < AC + CB$, $AB > AC - CB$.



PROOF. A straight line is the shortest distance between any two points (2, 1): hence $AB < AC + CB$. For the same reason, $AB + BC > AC$: hence $AB > AC - BC$ (AX. 5).

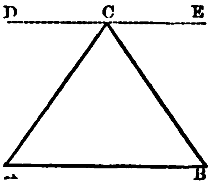
COR. In an isosceles triangle, one of the equal sides is greater than half the unequal side.

XVII.

Theorem. *In any triangle, the sum of the three angles is equal to two right angles.*

Hypoth. Let ABC be any triangle.

To be Proved. $\angle A + B + ACB = 2R$.



Proof. Through the vertex of either angle, as C, draw DE $\parallel$ AB, the opposite side.

Then $\angle A = ACD$ (10, 2),
and $\angle B = BCE$ (10, 2),
also $\angle ACB = ACB$:

hence $\angle A + B + ACB = ACD + BCE + ACB = 2R$ (4, 2).

Cor. 1. When two angles of a triangle are given, the third may be found by subtracting their sum from two right angles.

Cor. 2. If two angles of one triangle are respectively equal to two angles of another triangle, the third angles are also equal, and the triangles are mutually equiangular.

Cor. 3. Every triangle must have at least two acute angles; for, if two were obtuse or right angles, the sum of the three angles would be greater than two right angles.

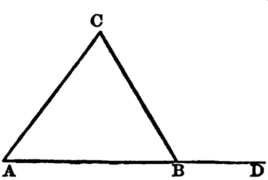
Cor. 4. In any right-angled triangle, the sum of the acute angles is equal to R.

Cor. 5. In an equiangular triangle, each angle is equal to $\frac{1}{3}$ of $2R = \frac{2}{3}R$.

XVIII.

Theorem. *If one side of a triangle be produced, the exterior angle is equal to the two interior angles not adjacent.*

Hypoth. In the triangle ABC, one side, AB, is produced, forming the exterior angle, CBD.



To be Proved. $\angle CBD = A + C$.

Proof. $\angle CBD$ and $\angle A + C$ are each supplements of the same angle, ABC (4 and 17): hence $\angle CBD = A + C$ (5 Cor.).

COR. 1. The exterior angle of a triangle is greater than either of the two interior angles not adjacent.

EXERCISE 1. Prove this proposition by an auxiliary line drawn through B, parallel to AC.

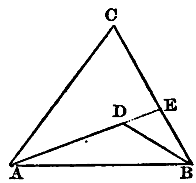
EXERCISE 2. To what is the sum of the three exterior angles of a triangle equal?

XIX.

Theorem. *If, from any point within a triangle, two straight lines be drawn to the extremities of either side,*

1. *Their sum will be less than the sum of the other two sides of the triangle.*

2. *The angle they form with each other will be greater than the angle between those sides.*



HYPOTH. From the point D, in the triangle ABC, the lines DA and DB are drawn to the extremities of the side AB.

TO BE PROVED. 1. $AD + DB < AC + CB$.

PROOF. Prolong AD until it meets CB in the point E.

In $\triangle AEC$, $AE < AC + CE$ (16).

To each add EB, and $AE + EB < AC + CB$.

In $\triangle DEB$, $DB < ED + EB$.

To each add AD, and $AD + DB < AE + EB$.

Much more is $AD + DB < AC + CB$.

TO BE PROVED. 2. $\angle ADB > ACB$.

PROOF. $\angle ADB > DEB$ (18, 1),

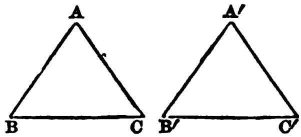
also $\angle DEB > ACB$ (18, 1):

much more is $\angle ADB > ACB$.

XX.

Theorem. *If two triangles have two sides and the included angle of one equal to two sides and the included angle of the other, each to each, the two triangles will be congruent.*

Hypoth. In the triangles ABC and A′B′C′, AB = A′B′, AC = A′C′, and ∠ A = A′.



To be Proved. ∠ B = B′, ∠ C = C′, BC = B′C′, or ABC ≅ A′B′C′.

Proof. The angle A may be laid upon its equal angle, A′, so that AB shall lie upon its equal side, A′B′, and AC upon its equal side, A′C.′

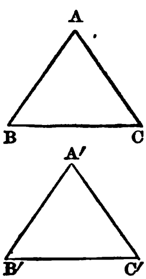
The points B and C will lie upon B′ and C′, and the line BC will coincide with B′C′ (2, 2): hence ABC ≅ A′B′C′.

XXI.

Theorem. *If two triangles have a side and two angles of the one equal to a side and two similarly-situated angles of the other, each to each, the triangles will be congruent.*

The two angles in each triangle may be both adjacent to, or one adjacent to, and the other opposite, the equal sides. But the two cases reduce to one, since the third angles will also be equal (17, 2).

Hypoth. BC = B′C′, ∠ B = B′, ∠ C = C′.



To be Proved. △ ABC ≅ A′B′C′.

Proof. The triangle ABC may be so applied to the triangle A′B′C′, that the line BC will fall upon its equal, B′C′, the point B on B′, and the point C on C′: BA will take the direction B′A′, since ∠ B = B′, and the point A will fall somewhere in the line B′A′: CA will take the direction C′A′, since ∠ C = C′, and the point A will fall somewhere in the line C′A′.

Hence the point A, falling at the same time in the two lines, B′A′ and C′A′, must fall at their intersection, A′, and the two triangles coincide throughout; or

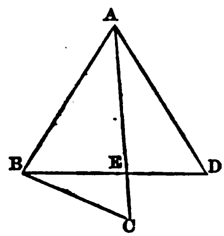
△ ABC ≅ A′B′C′.

XXII.

Theorem. *If two triangles have two sides of the one equal to two sides of the other, each to each, but the included angles unequal, the third sides will be unequal; and the greater third side will belong to the triangle which has the greater included angle.*

Hypoth. AB = AB, AC = AD, ∠BAD > BAC.

To be Proved. BD > BC.



Proof. Since ∠BAD > BAC, it follows that ABC + ACB > ABD + ADB (17); that is, ABC > ABD, or ACB > ADB.

Hence the triangles may be so applied to each other, that the side BC, opposite the smaller angle, BAC, shall lie without the triangle ABD.

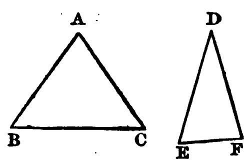
Then BE + EC > BC (16),
and ED + EA > AD (16),
adding these we have BD + AC > BC + AD:
hence BD > BC, since AC = AD (Hypoth.).

XXIII.

Theorem. *If two triangles have two sides of the one equal to two sides of the other, each to each, but the third sides unequal, the angle contained by the sides of that, which has the greater third side will be greater than the angle contained by the sides of the other.*

Hypoth. AB = DE, AC = DF, BC > EF.

To be Proved. ∠BAC > EDF.

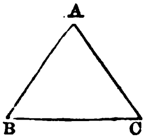


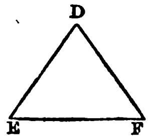
Proof. If ∠BAC is not > EDF, either ∠BAC = EDF, or ∠BAC < EDF. But if ∠BAC = EDF, then BC = EF (20); and if ∠BAC < EDF, then BC < EF

(22). These conclusions are both contrary to the hypothesis: hence $\angle$ BAC $>$ EDF.

XXIV.

Theorem. *If two triangles are mutually equilateral, they are also mutually equiangular and congruent.*





Hypoth. AB = DE, AC = DF, BC = EF.

To be Proved. $\angle$ A = D, $\angle$ B = E, $\angle$ C = F.

Proof. If $\angle$ A $\gtrless$ D, then BC $\gtrless$ EF;

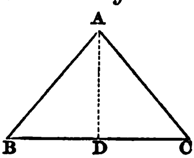
but BC = EF (Hypoth.):

hence $\angle$ A = D.

In the same manner, it may be proved that $\angle$ B = E, and $\angle$ C = F.

XXV.

Theorem. *The angles opposite the equal sides of an isosceles triangle are equal.*



Hypoth. AB = AC.

To be Proved. $\angle$ B = C.

Proof. Bisect $\angle$ BAC by the line AD;

then AB = AC (Hypoth.),

AD = AD,

and $\angle$ BAD = CAD, by construction:

hence $\triangle$ ABD $\cong$ ACD, and $\angle$ B = C (20).

Cor. 1. Since $\triangle$ ABD $\cong$ ACD, the side BD = CD, and $\angle$ ADB = ADC = R.

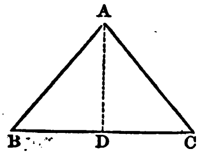
Hence *the line bisecting the angle between the equal sides of an isosceles triangle bisects the other side at right angles.*

Also a *line joining the vertex of the angle between the equal sides of an isosceles triangle, and the middle of the opposite side, bisects the angle, and is perpendicular to the side.*

Cor. 2. Every equilateral triangle is equiangular.

XXVI.

Theorem. Conversely, *if two angles of a triangle are equal, the sides opposite to them are also equal, and the triangle is isosceles.*



Hypoth. $\angle B = C$.

To be Proved. $AB = AC$.

Proof. Let the angle BAC be bisected by the line AD.

Then $\angle BAD = CAD$;

$\angle B = C$ (Hypoth.),

and $AD = AD$:

hence $\triangle ABD \cong ACD$ (21),

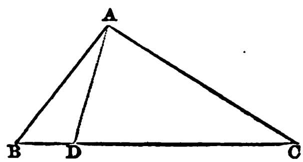
and $AB = AC$.

Cor. Every equiangular triangle is equilateral.

Exercise. If the distance from the vertex of an angle of a triangle to the middle of the opposite side is equal to half that side, the angle is a right angle, and conversely.

XXVII.

Theorem. *If two sides of a triangle are unequal, the angles opposite to them are also unequal; and the greater angle is opposite to the greater side.*



Hypoth. In the triangle ABC, the side $BC > AC$.

To be Proved. $\angle CAB > CBA$.

Proof. From BC cut off $DC = AC$, join AD.

Then $\angle CAD = CDA$ (25):

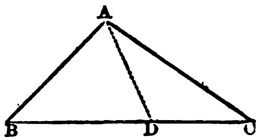
hence $\angle CAB > CDA$;

but $\angle CDA > CBA$ (18, 1):

much more is $\angle CAB > CBA$.

XXVIII.

Theorem. Conversely, *if two angles of a triangle are unequal, the sides opposite to them are also unequal; and the greater side is opposite the greater angle.*



Hypoth. $\angle CAB > CBA$.

To be Proved. $BC > AC$.

Proof. Draw the line AD, making $\angle DAB = DBA$.

Then $AD = BD$ (26),

and $AD + DC = BC$;

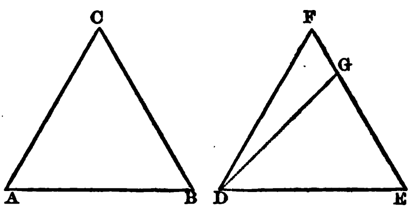
but $AD + DC > AC$ (16):

hence $BC > AC$.

Cor. In a right-angled or an obtuse-angled triangle, the greatest side is opposite the right angle or obtuse angle.

XXIX.

Theorem. *If two triangles have two sides of the one equal to two sides of the other, each to each, and the angle opposite the greater of these two sides in each equal, the triangles will be congruent.*



Hypoth. $AC = DF$, $AB = DE$, $AC > AB$, $DF > DE$, $\angle B = E$.

To be Proved. $\triangle ABC \cong DEF$.

Proof. It will be necessary only to prove that $BC = EF$ (20, or 24).

If BC and EF are not equal, one must be the less. Suppose $BC < EF$.

Cut off $EG = BC$; join DG;

then $\triangle DEG \cong ABC$ (20):

hence $DG = AC = DF$, and $\angle DGF = F$ (25);

but $\angle DGF > DEG$ (18, 1).

Hence $\angle F > DEG$, and $DE > DF$ (28), which is contrary to hypoth.: hence the supposition $BC < EF$ is false. In like manner, it may be shown that EF is not less than BC:

hence $BC = EF$ and $\triangle ABC \cong DEF$ (20 or 24).

Cor. Two right-angled triangles are congruent when any

two sides of the one are equal to two similarly-situated sides of the other, each to each. The same is true of two obtuse-angled triangles, when the obtuse angles are equal.

XXX.

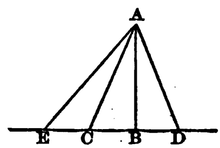
Theorem. *From a point without a straight line, —*

1. *But one perpendicular can be drawn to that line.*

2. *The perpendicular will be shorter than any oblique line.*

3. *Any two oblique lines that meet the line at equal distances from the foot of the perpendicular will be equal.*

4. *Of two oblique lines, that which meets the straight line farther from the foot of the perpendicular will be the greater.*



HYPOTH. 1. ED is a straight line, and A a point without it.

TO BE PROVED. But one perpendicular, AB, can be drawn from A to ED.

PROOF. For if two perpendiculars, AB and AC, could be drawn, the triangle ABC would contain two right angles, ABC and ACB; which is impossible (17, 3).

HYPOTH. 2. $AB \perp ED$, and AC is any oblique line.

TO BE PROVED. $AB < AC$.

PROOF. In the triangle ABC,

$$\angle ACB < ABC = R \ (17, 3):$$

hence $\qquad AB < AC$ (28).

HYPOTH. 3. $BC = BD$.

TO BE PROVED. $AC = AD$.

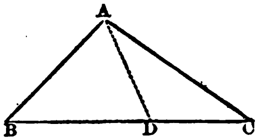
PROOF. $\qquad \triangle ABC \cong ABD$ (20):

hence $\qquad AC = AD$.

HYPOTH. 4. $BE > BC$.

TO BE PROVED. $AE > AC$.

PROOF. Since $\angle ACE$ is an exterior angle of the triangle ABC, we have $\angle ACE > ABC = R$ (18, 1):



Hypoth. $\angle CAB > CBA$.

To be Proved. $BC > AC$.

Proof. Draw the line AD, making $\angle DAB = DBA$.

Then $AD = BD$ (26),

and $AD + DC = BC$;

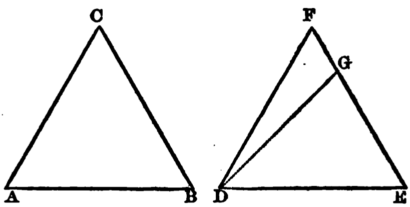
but $AD + DC > AC$ (16):

hence $BC > AC$.

Cor. In a right-angled or an obtuse-angled triangle, the greatest side is opposite the right angle or obtuse angle.

XXIX.

Theorem. *If two triangles have two sides of the one equal to two sides of the other, each to each, and the angle opposite the greater of these two sides in each equal, the triangles will be congruent.*



Hypoth. $AC = DF$, $AB = DE$, $AC > AB$, $DF > DE$, $\angle B = E$.

To be Proved. $\triangle ABC \cong DEF$.

Proof. It will be necessary only to prove that $BC = EF$ (20, or 24).

If BC and EF are not equal, one must be the less. Suppose $BC < EF$.

Cut off $EG = BC$; join DG;

then $\triangle DEG \cong ABC$ (20):

hence $DG = AC = DF$, and $\angle DGF = F$ (25);

but $\angle DGF > DEG$ (18, 1).

Hence $\angle F > DEG$, and $DE > DF$ (28), which is contrary to hypoth.: hence the supposition $BC < EF$ is false. In like manner, it may be shown that EF is not less than BC:

hence $BC = EF$ and $\triangle ABC \cong DEF$ (20 or 24).

Cor. Two right-angled triangles are congruent when any

two sides of the one are equal to two similarly-situated sides of the other, each to each. The same is true of two obtuse-angled triangles, when the obtuse angles are equal.

XXX.

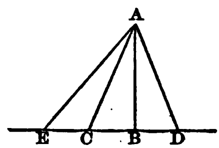
Theorem. *From a point without a straight line,—*

1. *But one perpendicular can be drawn to that line.*

2. *The perpendicular will be shorter than any oblique line.*

3. *Any two oblique lines that meet the line at equal distances from the foot of the perpendicular will be equal.*

4. *Of two oblique lines, that which meets the straight line farther from the foot of the perpendicular will be the greater.*



Hypoth. 1. ED is a straight line, and A a point without it.

To be Proved. But one perpendicular, AB, can be drawn from A to ED.

Proof. For if two perpendiculars, AB and AC, could be drawn, the triangle ABC would contain two right angles, ABC and ACB; which is impossible (17, 3).

Hypoth. 2. AB $\perp$ ED, and AC is any oblique line.

To be Proved. AB $<$ AC.

Proof. In the triangle ABC,

$$\angle ACB < ABC = R \ (17, 3):$$

hence $$AB < AC \ (28).$$

Hypoth. 3. BC $=$ BD.

To be Proved. AC $=$ AD.

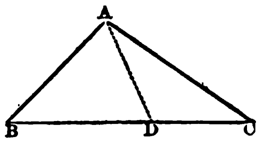
Proof. $$\triangle ABC \cong ABD \ (20):$$

hence $$AC = AD.$$

Hypoth. 4. BE $>$ BC.

To be Proved. AE $>$ AC.

Proof. Since $\angle$ ACE is an exterior angle of the triangle ABC, we have $\angle ACE > ABC = R$ (18, 1):



HYPOTH. $\angle$ CAB $>$ CBA.

TO BE PROVED. BC $>$ AC.

PROOF. Draw the line AD, making $\angle$ DAB $=$ DBA.

Then AD $=$ BD (26),

and AD $+$ DC $=$ BC;

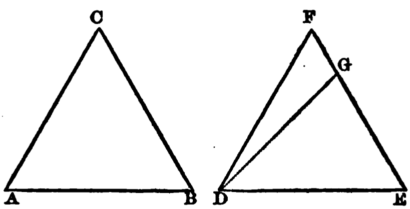
but AD $+$ DC $>$ AC (16):

hence BC $>$ AC.

COR. In a right-angled or an obtuse-angled triangle, the greatest side is opposite the right angle or obtuse angle.

XXIX.

Theorem. *If two triangles have two sides of the one equal to two sides of the other, each to each, and the angle opposite the greater of these two sides in each equal, the triangles will be congruent.*



HYPOTH. AC $=$ DF, AB $=$ DE, AC $>$ AB, DF $>$ DE, $\angle$ B $=$ E.

TO BE PROVED. $\triangle$ ABC $\cong$ DEF.

PROOF. It will be necessary only to prove that BC $=$ EF (20, or 24).

If BC and EF are not equal, one must be the less. Suppose BC $<$ EF.

Cut off EG $=$ BC; join DG;

then $\triangle$DEG $\cong$ ABC (20):

hence DG $=$ AC $=$ DF, and $\angle$ DGF $=$ F (25);

but $\angle$ DGF $>$ DEG (18, 1).

Hence $\angle$ F $>$ DEG, and DE $>$ DF (28), which is contrary to hypoth.: hence the supposition BC $<$ EF is false. In like manner, it may be shown that EF is not less than BC:

hence BC $=$ EF and $\triangle$ ABC $\cong$ DEF (20 or 24).

COR. Two right-angled triangles are congruent when any

two sides of the one are equal to two similarly-situated sides of the other, each to each. The same is true of two obtuse-angled triangles, when the obtuse angles are equal.

XXX.

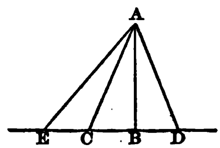
Theorem. *From a point without a straight line,—*

1. *But one perpendicular can be drawn to that line.*

2. *The perpendicular will be shorter than any oblique line.*

3. *Any two oblique lines that meet the line at equal distances from the foot of the perpendicular will be equal.*

4. *Of two oblique lines, that which meets the straight line farther from the foot of the perpendicular will be the greater.*



Hypoth. 1. ED is a straight line, and A a point without it.

To be Proved. But one perpendicular, AB, can be drawn from A to ED.

Proof. For if two perpendiculars, AB and AC, could be drawn, the triangle ABC would contain two right angles, ABC and ACB; which is impossible (17, 3).

Hypoth. 2. $AB \perp ED$, and AC is any oblique line.

To be Proved. $AB < AC$.

Proof. In the triangle ABC,

$$\angle ACB < ABC = R \ (17, 3):$$

hence $$AB < AC \ (28).$$

Hypoth. 3. $BC = BD$.

To be Proved. $AC = AD$.

Proof. $$\triangle ABC \cong ABD \ (20):$$

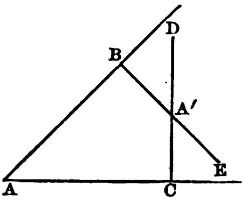
hence $$AC = AD.$$

Hypoth. 4. $BE > BC$.

To be Proved. $AE > AC$.

Proof. Since $\angle ACE$ is an exterior angle of the triangle ABC, we have $\angle ACE > ABC = R$ (18, 1):

For if $A'B \perp AB$, and $A'C \perp AC$, the sides of either angle at A′ are perpendicular to the sides of $\angle A$.



Then in the quadrilateral ABA′C, $\angle A$ is the supplement of $\angle BA'C$, or its equal, DA′E: hence $\angle BA'D$ and CA′E, being supplements of $\angle BA'C$, are equal to $\angle A$.

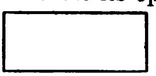
QUADRILATERALS.

XXXVI.

DEFINITIONS.

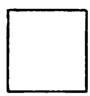
1. Quadrilaterals are of three kinds: *parallelograms*, *trapezoids*, and *trapeziums*.

2. A **parallelogram** is a quadrilateral that has its opposite sides parallel.



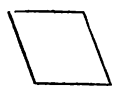
a. A **rectangle** is a right-angled parallelogram.

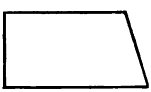
A **square** is an equilateral rectangle.



b. A **rhomboid** is an oblique-angled parallelogram.

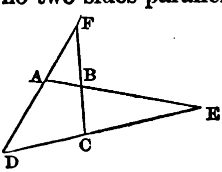
A **rhombus** is an equilateral rhomboid.





3. A **trapezoid** is a quadrilateral that has only one pair of its opposite sides parallel.

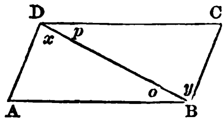
4. A **trapezium** is a quadrilateral that has no two sides parallel.



A **complete quadrilateral** is formed by producing the opposite sides of a trapezium until they meet. It has six angles and three diagonals, viz., AC, BD, and FE.

XXXVII.

Theorem. *The diagonal of a parallelogram divides it into two congruent triangles.*



Hypoth. DB is a diagonal of the parallelogram ABCD.

To be Proved. $\triangle$ DAB $\cong$ BCD.

Proof. DB = DB.

$\angle x = y$, since AD $\parallel$ BC (10, 2),

$\angle p = o$, since DC $\parallel$ AB (10, 2):

hence $\triangle$ DAB $\cong$ BCD (21).

Cor. 1. Since $\triangle$ DAB $\cong$ BCD, it follows (14, 6) that AB = CD, AD = BC, $\angle$ A = C, and $\angle x + p = o + y$: hence *the opposite sides and angles of a parallelogram are equal.*

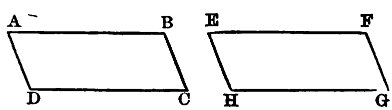
and $\angle$ EBA = EDC
hence $\triangle$ AEB $\cong$ CED
that is, AE = EC, a
Cor. 1. If AB = BC,

Cor. 2. Two parallels, AD and BC, included between two other parallels, AB and CD, are equal.

Cor. 3. If the parallels AD and BC are perpendicular to AB and DC, they will measure the distances of the points A and B from DC, and be equal: hence *two parallels are everywhere equally distant.*

Cor. 4. *If one angle of a parallelogram is a right angle, the three remaining angles will be right angles; also the two angles at the extremities of one side are supplements of each other.*

Cor. 5. *Two parallelograms which have two adjacent sides and the included angle of the one equal to two adjacent sides and an included angle of the other are congruent.*



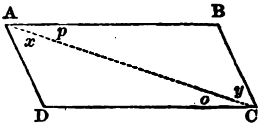
If AB = EF, BC = FG, and $\angle$ B = F, then will ABCD $\cong$ EFGH.

For it is evident, from the preceding corollaries, that $\angle$ C = G (Cor. 4), DC = HG, $\angle$ D = H, AD = EH (Cor. 1); and the parallelograms may

be applied to each other so as to correspond in all their parts.

XXXVIII.

Theorem. *If the opposite sides of a quadrilateral are equal, each to each, the figure is a parallelogram.*



HYPOTH. AB = CD, and AD = BC.

TO BE PROVED. AB || CD and AD || BC.

PROOF. Draw the diagonal AC.

Then △ ABC ≅ CDA (24);
that is ∠ $o = p$, hence AB || CD (11, 2);
and ∠ $x = y$, hence AD || BC (11, 2); that is, ADCB is a parallelogram (36, 2).

KINDS: *parallelograms, trape-*

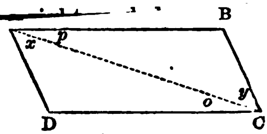
XXXI

Theorem. *If two opposite si*adrilateral *equal and parallel, the figure is a* el.

HYPOTH. AB = || DC.

TO BE PROVED. AD || BC.

PROOF. Draw the diagonal AC.

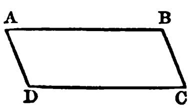


In the two triangles, ABC and CDA,

we have AB = DC (Hypoth.),
AC = AC,
and ∠ $o = p$, since AB || DC (10, 2):
hence △ ABC ≅ CDA (20):
consequently ∠ $x = y$, and AD || BC (11, 2).

XL.

Theorem. *If the opposite angles of a quadrilateral are equal, each to each, the figure is a parallelogram.*



HYPOTH. ∠ A = C, and ∠ B = D.

TO BE PROVED. AB || DC, and AD || BC.

PROOF. ∠ A + B + C + D = 4R (35, 3);
but ∠ A = C, and ∠ B = D (Hypoth.).

Substituting these values,

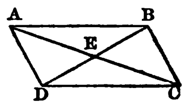
we have $2A + 2B = 4R$, or, $A + B = 2R$:

hence $AD \parallel BC$ (11, 3).

In like manner it may be proved that $AB \parallel DC$.

XLI.

Theorem. *The diagonals of a parallelogram bisect each other.*



HYPOTH. AC and BD are diagonals of the parallelogram ABCD.

TO BE PROVED. $AE = EC$, and $BE = ED$.

PROOF. In the triangles AEB and CED,

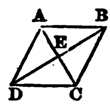
the side $AB = CD$ (37, 1),

$\angle BAE = DCE$ (10, 2),

and $\angle EBA = EDC$ (10, 2):

hence $\triangle AEB \cong CED$ (21);

that is, $AE = EC$, and $BE = ED$.

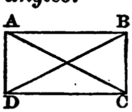


COR. 1. If $AB = BC$,

then $\triangle AEB \cong CEB$, being mutually equilateral (24):

hence $\angle AEB = BEC = R$; or, $BE \perp AC$ (3, 4).

Hence *the diagonals of a rhombus bisect each other at right angles.*

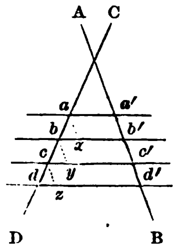


COR. 2. If the parallelogram ABCD is right angled, $\triangle ADC \cong BCD$: hence $AC = BD$; or, *the diagonals of a rectangle are equal.*

XLII.

Theorem. *If a series of parallels cutting any two straight lines intercept equal distances on one of these lines, they also intercept equal distances on the other line.*

HYPOTH. The two lines AB and CD are cut by the parallels, aa', bb', cc', dd', . . . intercepting on CD the distances $ab = bc = cd = \ldots$



TO BE PROVED. $a'b' = b'c' = c'd' = \ldots$

PROOF. Through the points $a, b, c, \ldots$ draw $ax \parallel a'b'$, $by \parallel b'c'$, $cz \parallel c'd'$, ...

Then $\triangle\, axb \cong byc \cong czd = \ldots$ (21):

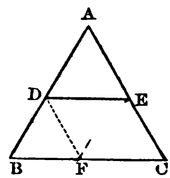
hence $ax = by = cz$, also

$aa'b'x$, $bb'c'y$, ... are parallelograms,

and $ax = a'b'$, $by = b'c'$, ... (37, 1):

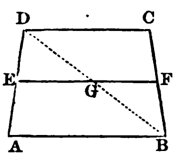
hence $a'b' = b'c' = c'd' = \ldots$

COR. 1. If in the triangle ABC, AD = DB, and DE $\parallel$ BC, then AE = EC. Also since two points, D and E, fix the position of a straight line, it follows, that if AD = DB, and AE = EC, then DE $\parallel$ BC; that is, *the line joining the middle points of two sides of a triangle is parallel to the third side.*



COR. 2. Since DE = BF = FC, DE $= \frac{BC}{2}$; that is, *the line joining the middle points of two sides of a triangle is equal to one-half the third side.*

COR. 3. If, in the trapezoid ABCD, E and F are the middle points of the non-parallel sides, it follows that EF $\parallel$ AB. Also draw the diagonal DB; then, by Cor. 2,



$$EG = \frac{AB}{2}, \text{ and } GF = \frac{DC}{2}:$$

hence
$$EG + GF = \frac{AB + DC}{2},$$

or
$$EF = \frac{AB + DC}{2}.$$

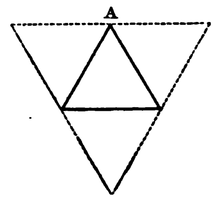
That is, *the line joining the middle points of the non-parallel sides of a trapezoid is parallel to the other two sides, and is equal to one-half their sum.*

EXERCISES.

1. In an oblique-angled parallelogram, the diagonal joining the vertices of the smaller angles is the greater.

2. The diagonals of a rhombus bisect its angles.

3. The lines joining the middle points of the sides of a triangle divide it into four congruent triangles.



4. The triangle formed by lines drawn through the vertices of the angles of any triangle parallel to the opposite sides is divided into four congruent triangles.

5. If the line bisecting an angle of a triangle bisects also the opposite side, it is perpendicular to that side, and the triangle is isosceles.

Prolong the bisecting line AD until DE $=$ AD; join, &c.

6. If the base of an isosceles triangle be produced, the exterior angle, minus half the vertical angle, is equal to one right angle.

7. If BC is the base of an isosceles triangle, ABC, and BD is drawn perpendicular to AC, the angle DBC $= \frac{1}{2}$BAC.

8. The base BC of an isosceles triangle, ABC, is parallel to a line bisecting the exterior angle at A.

9. If, from a variable point in the base of an isosceles triangle, lines be drawn parallel to the equal sides, the perimeter of the parallelogram thus formed is constant.

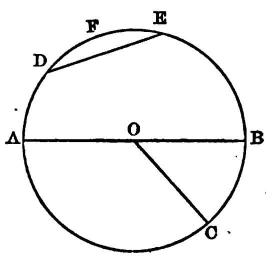
10. Any line bisecting the diagonal of a parallelogram bisects the parallelogram.

BOOK II.

THE CIRCLE.

I.—DEFINITIONS.

1. A **circle** is a portion of a plane bounded by a curved line every point of which is equally distant from a point within called the **centre**.



2. The curve is called the **circumference** of the circle.

3. An **arc** is a portion of the circumference; as, AD.

4. A **radius** is a line drawn from the centre to the circumference; as, OC or OA.

5. A **diameter** is a line passing through the centre, and terminated each way by the circumference; as, AB.

COR. 1. A radius is one-half a diameter: all radii of the same circle are equal; and they measure the distance of the centre from the circumference.

COR. 2. A point at a greater distance from the centre is without the circle, and a point at a less distance is within the circle.

Hence the circumference is the **locus** of all the points of a plane that are equally distant from a given point.

COR. 3. Circles with equal radii are congruent; for, if we apply the centres to each other, the circumferences will coin-

cide: otherwise some points would be at a greater or less distance from the centre than the radius.

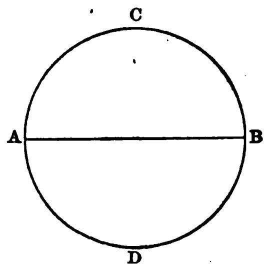
6. A **chord** is a line joining any two points of the circumference; as, DE.

7. A **segment** is the part of the circle included between an arc and its chord; as, DFE.

8. A **sector** is the figure included between an arc and the radii drawn to its extremities; as, BOC.

II.

Theorem. *A diameter divides the circle and its circumference into two congruent parts.*



HYPOTH. AB is a diameter of the circumference ACBD.

TO BE PROVED. ACB ≅ ADB.

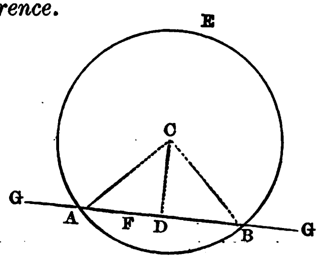
PROOF. If the part ACB be turned about the diameter AB, the arc ACB will coincide with ADB, since every point of the circumference is equally distant from the centre.

Hence ACB ≅ ADB.

DEF. The segment ACB is called a *semicircle;* and the arc ACB, a *semi-circumference.*

III.

Theorem. *A straight line cannot have more than two points common with a circumference.*



HYPOTH. AB is any straight line cutting the circumference AEB.

TO BE PROVED. AB can meet AEB in only two points, A and B.

PROOF. Draw the radii, CA and CB.

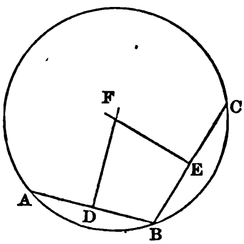
Any point, F, between A and B, is nearer the centre, C

(I., 30), and is within the circumference (1, Cor. 2). Any point, G, in AB extended, is farther from the centre, C (I., 30), and is without the circumference (1, Cor. 2). Hence A and B are the only two points of the line AB common with the circumference.

Cor. The circumference of a circle cannot be made to pass through three points which lie in the same straight line.

IV.

Theorem. *Through any three points not in the same straight line, one circumference may be made to pass, and but one.*



Hypoth. A, B, and C are any three points not in the same straight line.

To be Proved. One circumference, and only one, may be made to pass through A, B, and C.

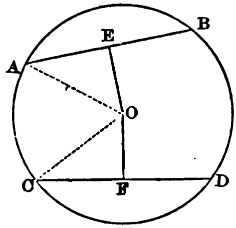
Proof. Draw AB and BC. From D and E, the middle point of these lines, erect the perpendiculars, DF and EF. They will meet in a point, F (I., 13). DF is the locus of all the points equally distant from A and B; EF is the locus of all the points equally distant from B and C: consequently their intersection, F, is equally distant from A, B, and C, and is the only point. Hence the circumference drawn from the centre, F, with a radius FA = FB = FC, will pass through the points A, B, and C, and is the only one that can be drawn through those points.

Cor. The centre of a circle lies in the perpendicular drawn from the middle of a chord. Hence *the perpendicular from the centre of a circle to a chord bisects that chord.*

V.

Theorem. *Equal chords are equally distant from the centre of a circle; and, of two unequal chords, the less is more remote from the centre.*

HYPOTH. 1. In the circle whose centre is O, the chord AB = CD.



Draw the perpendiculars, OE and OF. They measure the distance of the chords AB and CD from the centre, O.

TO BE PROVED. OE = OF.

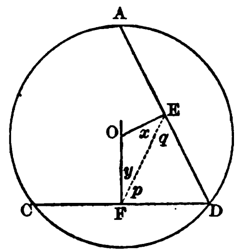
PROOF. Draw the radii, OA and OC. Then in the right-angled triangles, OAE and OCF, the side OA = OC,

and AE = CF, being halves of equal chords:

hence OE = OF (I., 29, Cor.).

HYPOTH. 2. The chord CD < DA, OF $\perp$ CD, OE $\perp$ AD.

TO BE PROVED. OF > OE.



PROOF. Suppose the chords turned until they have the position CD and DA. Draw EF. Then, in the triangle EFD, the side FD < ED, being halves of unequal chords.

Hence $\angle q < p$ (I., 27):

but $\angle$OED = OFD, being right angles;

hence $\angle x > y$, and OF > OE (I., 28).

COR. The diameter is the greatest chord.

EXERCISE 1. State and prove the converse of this proposition.

EXERCISE 2. The shortest chord that can be drawn through a given point within a circle is a chord perpendicular to the radius passing through that point.

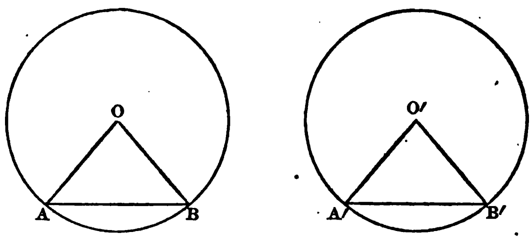
VI.

Theorem. *In the same or equal circles:—*

1. *To equal angles at the centre belong equal chords and equal arcs.*

2. *To equal chords belong equal angles at the centre, and equal arcs.*

3. *To equal arcs belong equal angles at the centre, and equal chords.*



Suppose the circles O and O′ equal.

Hypoth. 1. $\angle$ AOB = A′O′B′.

To be Proved. Chord AB = A′B′, and arc AB = A′B′.

Proof. $\triangle$ AOB $\cong$ A′O′B′ (I., 20).

Hence chord AB = A′B′.

If the circles be applied to each other so that the radius OA lie upon O′A′, the circumferences will coincide (1 Cor. 3), the triangle AOB will fall upon A′O′B′, the point B upon B′, and the arc AB will coincide with A′B′.

Hence arc AB = A′B′.

Hypoth. 2. Chord AB = A′B′.

To be Proved. $\angle$ AOB = A′O′B′, and arc AB = A′B′.

Proof. $\triangle$ AOB $\cong$ A′O′B′ (I., 24).

Hence $\angle$ AOB = A′O′B′;

and arc AB = A′B′ (part 1 of this proposition).

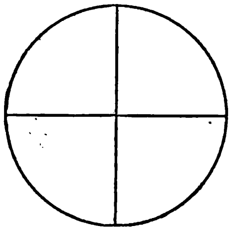
Hypoth. 3. Arc AB = A′B′.

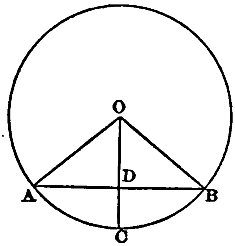
To be Proved. Chord AB = A′B′, and $\angle$AOB = A′O′B′.

Proof. If the equal arcs be applied to each other, so that the point A fall upon A′, and B upon B′, the chord AB will coincide with A′B′, since but one straight line can be drawn between two points: hence chord AB = A′B′. From this it follows that $\angle$ AOB = A′O′B′ (part 2).

Cor. 1. In the same or equal circles, sectors that have equal centre angles, equal chords, or equal arcs, are congruent.

COR. 2. Two diameters that are perpendicular to each other divide the circle and its circumference into four equal





parts. Each part of the circumference is called a *quadrant.*

COR. 3. Draw the radius OC $\perp$ AB. OC bisects the angle AOB (I., 25) ; or $\angle$ AOC = BOC : hence arc AC = BC. Also AD = DB (I., 25, Cor. 1). That is, *the radius drawn perpendicular to a chord bisects the chord and its arc.*

VII. — DEFINITIONS.

1. The **ratio** of two magnitudes is the quotient obtained by dividing the numerical representative of the one by that of the other; as, the ratio of A to B is $\frac{A}{B}$.

2. Two magnitudes are *commensurable* when a third may be found that is contained an exact number of times in each. This third magnitude is called the *common measure.* When two magnitudes have no common measure, they are *incommensurable*, and their ratio is called an *incommensurable ratio.*

Thus if a third line, E, be contained exactly 5 times in AB, and 3 times in CD, the lines AB and CD are commensurable, and their ratio is expressed by the fraction $\frac{5}{3}$; and, generally, if one magnitude be divided into n equal parts, and a second magnitude contain exactly m of those parts, their ratio will be $\frac{m}{n}$.

3. If, however, the second magnitude contains m of those parts, with a remainder less than the common measure, the fraction $\frac{m}{n}$ will be an *approximate value* of the ratio.

But the difference between $\frac{m}{n}$ and the true ratio will be less than $\frac{1}{n}$. Now, the number of parts, n, may be indefinitely increased by diminishing the common measure, E; and the fraction $\frac{1}{n}$ will then be indefinitely diminished.

Hence *an approximate ratio of two incommensurable magnitudes may be found, which will differ from the true ratio less than any assignable quantity, however small.*

4. **Measurement** consists in finding how often one quantity which is taken as the *unit* is contained in another quantity.

VIII.

Theorem. *Incommensurable ratios are equal, if their approximate ratios remain equal when the common measure is indefinitely diminished.*

Hypoth. $\frac{A}{B}$ and $\frac{C}{D}$ are two incommensurable ratios, which have the same approximate value, $\frac{m}{n}$, when the common measure is indefinitely diminished.

To be Proved. $\frac{A}{B}=\frac{C}{D}$.

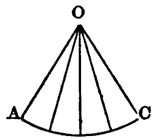
Proof. The ratios $\frac{A}{B}$ and $\frac{C}{D}$ exceed $\frac{m}{n}$ by a quantity less than $\frac{1}{n}$: hence the difference of $\frac{A}{B}$ and $\frac{C}{D}<\frac{1}{n}$.

But, by diminishing the common measure, $\frac{1}{n}$ may be made less than any assignable quantity, or zero: hence $\frac{A}{B}=\frac{C}{D}$.

IX.

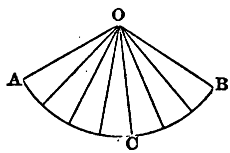
Theorem. *In the same or equal circles, angles at the centre have the same ratio as their intercepted arcs.*

HYPOTH. AOC and AOB are the angles at the centre of two equal circles. AC and AB are their intercepted arcs.



TO BE PROVED. $\dfrac{\angle \text{AOB}}{\angle \text{AOC}} = \dfrac{\text{arc AB}}{\text{arc AC}}$.

PROOF 1. Suppose the arcs commensurable. For example, let AB contain the common measure 7 times, and AC contain it 4 times.



Then $\dfrac{\text{arc AB}}{\text{arc AC}} = \dfrac{7}{4}$.

Draw radii to the points of division.

The centre angles AOB and AOC are respectively divided into 7 and 4 angles, which are equal, since they belong to equal arcs (6).

Hence $\dfrac{\angle \text{AOB}}{\angle \text{AOC}} = \dfrac{7}{4}$,

and $\dfrac{\angle \text{AOB}}{\angle \text{AOC}} = \dfrac{\text{arc AB}}{\text{arc AC}}$.

2. Suppose the arcs incommensurable. Let AC be divided into n equal parts, and suppose AB contains m of those parts, with a remainder less than one of the parts. To the points of division draw radii. They will divide the angle AOC into n equal parts, and AOB into m of these parts, with a remainder less than one of the parts. Hence $\dfrac{m}{n}$ will be the approximate ratio of $\dfrac{\angle \text{AOB}}{\angle \text{AOC}}$ and $\dfrac{\text{arc AB}}{\text{arc AC}}$ for any value of n; that is however small the common measure.

Consequently $\dfrac{\angle \text{AOB}}{\angle \text{AOC}} = \dfrac{\text{arc AB}}{\text{arc AC}}$ (8).

SCHOLIUM. Since an arc and its centre angle increase and

decrease in the same ratio, the same number may express the magnitude of both.

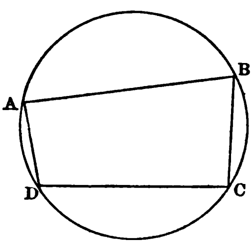
The unit of arc agreed upon is $\frac{1}{360}$ part of a circumference; and the centre angle of this arc is the unit of angle.

The unit of angle is, therefore, $\frac{1}{360}$ of 4R, since all the angles about the centre equal four right angles. The unit of arc and the unit of angle are both called a degree, and are denoted by the symbol °. $\frac{1}{60}$ part of a degree is called a *minute;* and $\frac{1}{60}$ of a minute, a *second.*

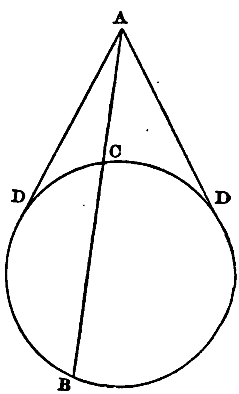
They are denoted respectively by the symbols ′ and ″; as, 60° 30′ 50″ is read 60 degrees, 30 minutes, and 50 seconds.

A right angle, or a quadrant, contains 90°.

X.—DEFINITIONS.



1. An angle is said to be *inscribed* when its sides are chords, and its vertex is in the circumference; as, ∠ ABC.



2. A polygon is *inscribed* when all its angles are inscribed. The circle is said to be *circumscribed* about the polygon.

3. A **secant** is a straight line that cuts the circumference in two points.

4. If a secant, AB, be turned about a fixed point, A, without the circle, in either direction, the two cutting points, C and B, will fall together in a point, D. The secant in this position is called a **tangent.** The point D is called the *point of contact.*

COR. 1. A tangent meets the circumference in but one point. All other points of the tangent lie without the circumference.

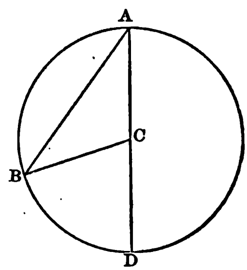
COR. 2. From a point without a circle, two tangents may be drawn to it, and only two; for the secant turned about this fixed point will be tangent in only two positions. In all other positions, it will either cut the circumference in two points, or lie entirely without it.

5. A polygon is *circumscribed* about a circle when all its sides are tangents. The circle, in that case, is said to be *inscribed* in the polygon.

XI.

Theorem. *The angle at the centre of a circle is double the inscribed angle upon the same arc.*

HYPOTH. BCD is an angle at the centre, and BAD an inscribed angle upon the same arc, BD.



There may be three cases.

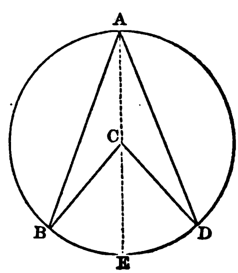
CASE 1. One side, AD, passes through the centre, C.

TO BE PROVED. $\angle BCD = 2BAD$.

PROOF. Since $BC = CA$, being radii, it follows that $\angle B = BAD$ (I., 25).

But BCD is an exterior angle of the triangle ABC.

Hence $\angle BCD = B + BAD = 2BAD$ (I., 18).



CASE 2. The centre, C, falls within the angle BAD.

TO BE PROVED. $\angle BCD = 2BAD$.

PROOF. Draw the diameter, AE.

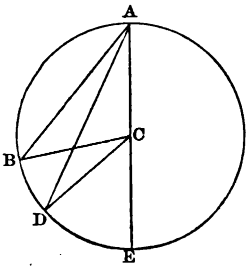
Then $\angle BCE = 2BAE$ (Case 1),

and $\angle DCE = 2DAE$ (Case 1).

Adding these two equations, we have, $\angle BCD = 2BAD$.

CASE 3. The centre, C, lies without the angle BAD.

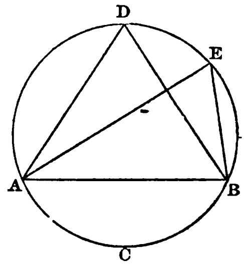
To be Proved. $\angle BCD = 2BAD$.



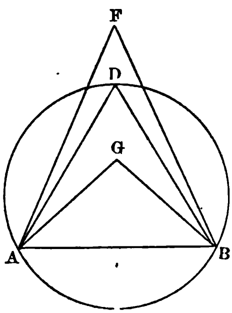
Proof. Draw the diameter ACE. Then $\angle BCE = 2BAE$ (Case 1), and $\angle DCE = 2DAE$ (Case 1).

Subtracting these two equations, we have $BCD = 2BAD$.

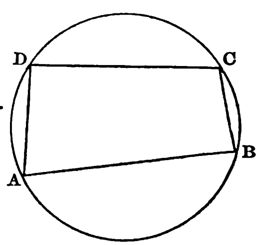
Cor. 1. An inscribed angle has the same numerical measure as half the arc included between its sides (9, Schol.).



Cor. 2. All the angles, ADB, AEB, inscribed in the same segment, are equal; for they have the same arc, ACB, included between their sides. Also the arc ADEB is the locus of the vertices of all the angles equal to ADB, whose sides pass through A and B; for if a vertex, F, be without the arc ADEB, the angle AFB $<$ ADB; and if a vertex, G, be within the arc, the angle AGB $>$ ADB (I., 19).

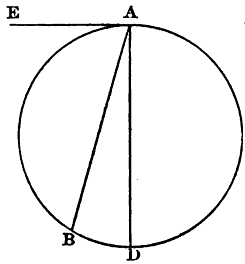


Cor. 3. An angle inscribed in a semicircle is a right angle; an angle inscribed in a segment less than a semicircle is an obtuse angle; and an angle inscribed in a segment greater than a semicircle is an acute angle.



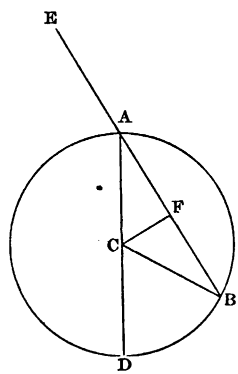
Cor. 4. In an inscribed quadrilateral, ABCD, two opposite angles, A and C, are together equal to two right angles, since their numerical measure is the same as one-half the entire circumference, or 180°.

COR. 5. If the side AB of the angle BAD be turned about the point A, so that the arc BD will constantly increase, the point B will approach A, and finally coincide with it. The side AB will then have the position AE, and be tangent to the circumference (10, 4). The angle BAD will become the angle EAD; and its corresponding arc, BD, will at the same time increase to ABD: that is, *the angle formed by a tangent and a chord meeting at the point of contact has the same numerical measure as half the intercepted arc.*



COR. 6. If the chord, AD, is a diameter, the arc ABD contains 180°: hence $\angle EAD = 90° = R$; or, *a tangent is perpendicular to a diameter drawn through the point of contact.*

COR. 7. If the line EA be turned about the point A, the radius CA will be an oblique line, and longer than a perpendicular, CF (I., 30): hence the point F will be within the circle (1, 6), and EA will cut the circumference in two points, A and B, at equal distances from F; that is, *any oblique line drawn through the extremity of a radius cuts the circumference in two points.* Also it is evident, that *a line drawn perpendicular to a radius, or diameter, at its extremity, is tangent to the circle.*



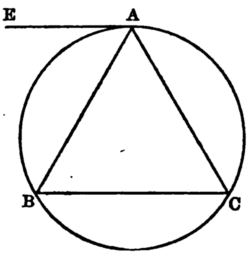
COR. 8. *The perpendicular drawn from the centre of a circle to a tangent will meet it at the point of contact.*

Also the perpendicular erected upon a tangent at the point of contact passes through the centre of the circle.

COR. 9. After the point B has passed the point A in the

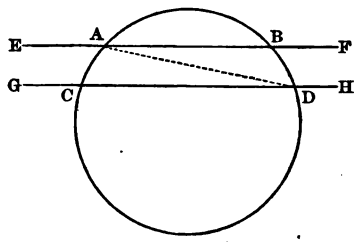
revolution of AB, the angle and arc continue to increase in the same ratio; and the angle DAE has the same numerical measure as half the arc DAB. This is also evident from the preceding proposition, since the supplement, ∠DAB, has the same numerical measure as half the arc DB, the remainder of the circumference.

Cor. 10. If EA is a tangent, and AB a chord drawn from the point of contact, A, also ACB an inscribed angle in the segment ACB, the angles EAB and ACB have the same numerical measure as half the arc AB: therefore ∠ EAB = ACB; that is, *the angle formed by a tangent and chord meeting at the point of contact, is equal to the angle inscribed in the opposite segment.*



XII.

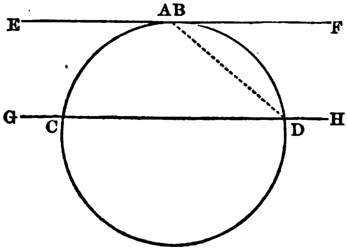
Theorem. *Two parallels intercept equal arcs on a circumference.*



Hypoth. The two parallels, EF and GH, intercept the arcs AC and BD.

To be Proved. Arc AC = BD.

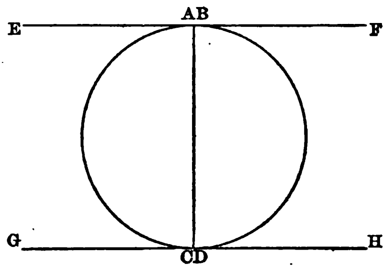
Proof. Draw AD. Then ∠FAD = ADG, being alternate angles (I., 10): hence the arcs AC and DB, included between the sides of those angles, have the same numerical measure, and are equal.



That is, arc AC = BD.

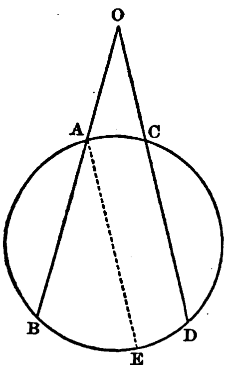
Cor. When the parallels

are tangents, the two points A and B, and also C and D, fall together in the points of contact: hence, when two tangents are parallel, their points of contact divide the circumference into two equal parts; and the line joining their points of contact is a diameter.



XIII.

Theorem. *When two secants cut each other* $\left[\begin{smallmatrix}\textit{without}\\ \textit{within}\end{smallmatrix}\right]$ *a circle, their angle has the same numerical measure as half the* $\left[\begin{smallmatrix}\textit{difference}\\ \textit{sum}\end{smallmatrix}\right]$ *of the included arcs.*

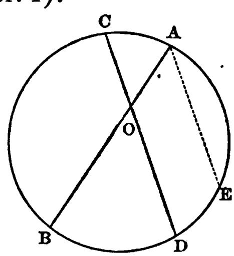


Hypoth. The two secants BA and DC cut each other $\left[\begin{smallmatrix}\text{without}\\ \text{within}\end{smallmatrix}\right]$ the circle in the point O.

To be Proved. ∠BOD has the same numerical measure as half the $\left[\begin{smallmatrix}\text{difference}\\ \text{sum}\end{smallmatrix}\right]$ of the arcs BD and CA.

Proof. Draw AE ∥ CD. Then ∠BAE = BOD (I., 10), and ∠BAE has the same numerical measure as half the arc BE (11, Cor. 1).

But arc BE = BD — ED = BD — CA, since AE ∥ CD (12), (Fig. 1); and BE = BD + DE = BD + CA (Fig. 2).



Hence ∠BOD has the same numerical measure as half the $\left[\begin{smallmatrix}\text{difference}\\ \text{sum}\end{smallmatrix}\right]$ of the arcs BD and CA.

Cor. In Fig. 1 let the two lines BO and DO turn together about the points B and D. The arc CA will constantly de-

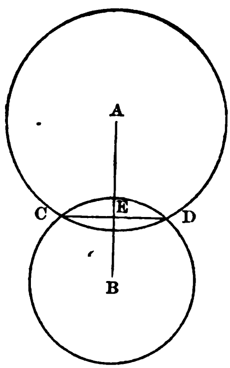
crease, and become zero when the point O falls upon the circumference, and the angle BOD is an inscribed angle: then will BD — CA = BD.

If the lines continue to turn, the point O will fall within the circle, and the arc CA will become less than zero, or negative: then will BD + CA become BD — AC. Hence, *when two lines intersect each other, their angle has the same numerical measure as half the algebraic difference of the included arcs of any circumference which they cut.*

RELATIVE POSITION OF CIRCLES.

XIV.

Theorem. *If two circumferences intersect each other, the line joining their centres is perpendicular to the common chord at its middle point.*



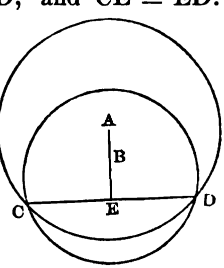
Hypoth. CD is a common chord of the circumferences whose centres are A and B.

To be Proved. AB ⊥ CD, and CE = ED.

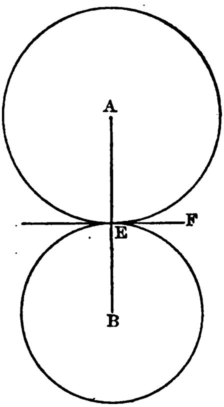
Proof. The centres, A and B, are equally distant from the extremities of the chord CD, and therefore fix the position of the perpendicular to the middle point of that chord (I., 31, Cor. 2).

Hence AB ⊥ CD, and CE = ED.

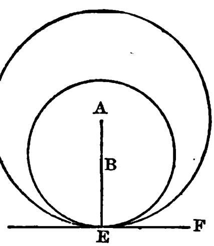
Cor. 1. Draw the radii AD and BD. Then, in the triangle ADB, the side AB < AD + BD, and AB > AD — BD (I., 16). Hence, *when two circumferences cut each other, the distance between their centres is less than the sum, and greater than the difference, of their radii.*



COR. 2. Let the circles be so moved, that the points C and D shall approach each other while the centres remain in the line AB. The cutting points, C and D, will be equally distant from the line AB, however near they are to each other; and they will fall together on that line. The circumferences will then touch each other, externally or internally, in the point E.

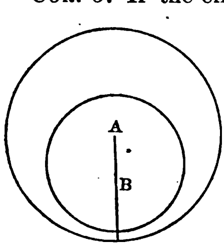


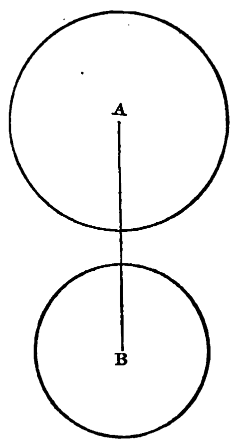
Hence, *If two circumferences touch each other externally or internally, their centres and point of contact are in the same straight line.*



COR. 3. Also, *if two circumferences touch each other* $\left[\begin{smallmatrix}\textit{externally}\\ \textit{internally}\end{smallmatrix}\right]$, *the distance of their centres is equal to the* $\left[\begin{smallmatrix}\textit{sum}\\ \textit{difference}\end{smallmatrix}\right]$ *of their radii.*

COR. 4. The line EF, drawn perpendicular to the radii AE and BE, at their extremity, is a common tangent to the two circumferences (11, Cor. 7).





COR. 5. If the circles be moved still farther in the same direction, they will be wholly $\left[\begin{smallmatrix}\text{without}\\ \text{within}\end{smallmatrix}\right]$ each other; and *the distances of their centres will be* $\left[\begin{smallmatrix}\textit{greater}\\ \textit{less}\end{smallmatrix}\right]$ *than the* $\left[\begin{smallmatrix}\textit{sum}\\ \textit{difference}\end{smallmatrix}\right]$ *of their radii.*

COR. 6. Conversely, it is evident that two circles may have five different relative positions depending upon the distance between their centres: —

1st, When the distance of their centres is greater than the sum of their radii, they are wholly exterior to each other (Cor. 5).

2d, When this distance is equal to the sum of their radii, the circumferences touch each other externally (Cor. 3).

3d, When this distance is less than the sum, and greater than the difference, of their radii, the circumferences cut each other (Cor. 1).

4th, When this distance is equal to the difference of their radii, the circumferences touch each other internally (Cor. 3).

5th, When this distance is less than the difference of their radii, one circumference is wholly within the other (Cor. 5).

Def. When two circles have a common centre, they are said to be **concentric.**

INSCRIBED AND CIRCUMSCRIBED POLYGONS.

XV.

Theorem. *A circle may be circumscribed about, and inscribed within, any triangle.*

This follows from I., 33, 34, and 11, Cor. 7.

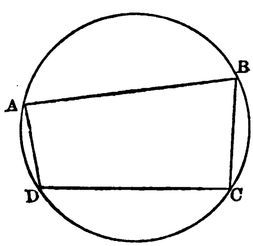
XVI.

Theorem. *The opposite angles of an inscribed quadrilateral are supplements of each other.*

This was proved in 11, Cor. 4.

XVII.

Theorem. Conversely, *a quadrilateral may be circumscribed by a circle, if two of its opposite angles are supplements of each other.*



Hypoth. $\angle D + B = 2R$.

To be Proved. The vertex B lies in the circumference that passes through the three points, A, D, and C (4).

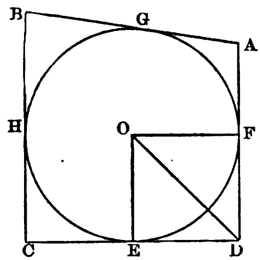
Proof. The arc ABC is the locus of the vertices of all angles that are supplements of the angle D (11, Cor. 2).

Hence the vertex B must lie in that arc, and the quadrilateral is circumscribed by the circle ABCD.

XVIII

Theorem. *In any quadrilateral circumscribing a circle, the sums of the two pairs of opposite sides are equal.*

HYPOTH. The quadrilateral ABCD is circumscribed about a circle.



TO BE PROVED. AD + BC = AB + DC.

PROOF. From the centre, O, draw lines to the points of contact, E and F, and the vertex, D. The right-angled triangles, OED and OFD, are equal (I., 29, Cor.).

Hence DE = DF.

In like manner it may be shown that AG = AF, BG = BH, CE = CH.

Adding the corresponding terms of these equations, we have, DE + AG + BG + CE = DF + AF + BH + CH;

or, AB + DC = AD + BC.

COR. The two tangents, DF and DE, drawn from any point, D, to a circumference, are equal.

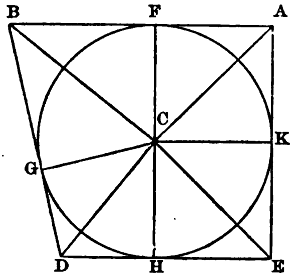
XIX.

Theorem. *A circle may be inscribed in a quadrilateral, when the sums of the two pairs of opposite sides are equal.*

HYPOTH. In the quadrilateral ABDE, AB + DE = BD + AE.

TO BE PROVED. A circle that touches the sides AE, AB, and BD, will also touch the fourth side, DE.

Proof. Bisect the two angles A and B by the lines AC and BC, and from the point of intersection, C, draw CK, CF, and CG, perpendicular to the sides AE, AB, and BD.



It is evident that CK = CF = CG (I., 21).

Hence a circle drawn from C as a centre, with a radius CK, will be tangent to AE, AB, and BD (11, Cor. 7). It will also be tangent to DE.

For draw CE, CD, and CH $\perp$ DE:

then $\quad AB + DE = BD + AE$ (Hypoth.),

and $\quad AB = BG + AK.$

Subtracting, we have, $DE = GD + KE.$

Turn the triangles CGD and CKE about the lines CD and CE, and apply them to the triangle CED, so that GD and KE will correspond with the equal side DE.

The equal perpendiculars, CG and CK, will coincide; and the triangles CGD and CKE will together form a triangle congruent with CED, since three sides of one will be equal to three sides of the other.

Hence the perpendicular CH = CG, and the circumference KFG will pass through the point H, and be tangent to DE; that is, a circle may be inscribed in the quadrilateral ABDE.*

PROBLEMS.

The following geometrical constructions require the aid of a ruler and compasses, with the use of which the student should become familiar. It is very important that he construct all the figures with care.

* This is the only simple, direct demonstration of this proposition known to the author, and was first given by him. (See Chauvenet.)

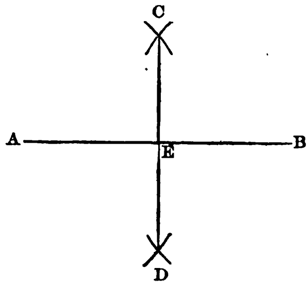
XX.

Problem. *To bisect a given straight line.*

Let AB be the given straight line.

It is required to find its middle point.

SOLUTION. From A and B as centres, with a radius greater than one-half of AB, describe arcs intersecting at C and D.



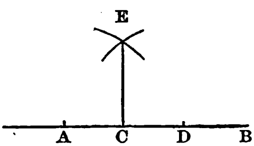
Draw the line CD: it will bisect AB in the point E.

PROOF. The points C and D are equally distant from the extremities A and B, and fix the position of the perpendicular through the middle point, E (I., 31, Cor. 2).

XXI.

Problem. *At a given point in a given straight line to draw a perpendicular to that line.*

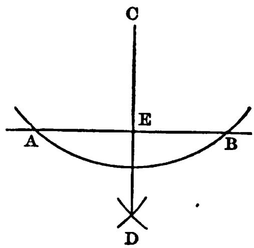
Let C be the given point in the given straight line AB.



SOLUTION. Take two points, A and D, in the given line at equal distances from C. From A and D as centres, with a radius greater than AC, describe two arcs intersecting at E. Draw EC. It will be perpendicular to AB, since the two points E and C are equally distant from A and D (I., 31, Cor. 2).

XXII.

Problem. *To draw a perpendicular to a straight line from a given point without that line.*



Let AB be the given line, and C the point without that line. It is required to draw a perpendicular from C to AB.

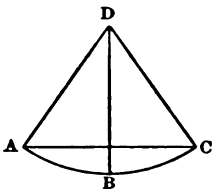
SOLUTION. From C as a centre, draw an arc that will cut AB in the

two points A and B. From A and B, with a radius greater than one-half AB, describe two arcs intersecting at D. Draw CD: it will be perpendicular to AB; for the two points C and D are equally distant from A and B (I., 31, Cor. 2).

XXIII.

Problem. *To bisect a given arc or a given angle.*

1st, Let ABC be the given arc.



Draw the radius DB perpendicular to the chord AC. It will bisect the arc in the point B (6, Cor. 3).

2d, Let ADC be the given angle.

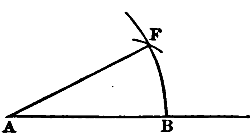
From D as a centre, with any radius, draw an arc, ABC, cutting the sides of the angle in A and C. Bisect the arc ABC by the line DB: then will DB bisect the angle ADC. For, since arc AB = BC, the angle ADB = BDC (6, Cor. 3).

Scholium. In the same manner, each half of an arc or angle may be bisected; and by successive bisections an arc or angle may be divided into 4, 8, 16, 32, &c., equal parts.

XXIV.

Problem. *At a given point in a given straight line to construct an angle equal to a given angle.*

Let A be the point in the given straight line AB, and C the given angle. It is required to construct an angle at A that shall be equal to C.



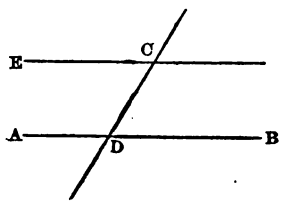
Solution. From C as a centre, with any radius, CD, draw an arc, DE, between the sides of the angle C. From A as a centre, with the same radius, draw an indefinite arc, BF. From B as a centre, with a radius equal to the chord of DE, ...ing BF in F. Draw AF. The angle

FAB is the angle required; for the arcs DE and BF have equal radii and equal chords, and are, therefore, equal: hence the angles A and C are equal (6).

XXV.

Problem. *Through a given point to draw a line parallel to a given straight line.*

Let C be the given point, and AB the given straight line. It is required to draw through C a line parallel to AB.

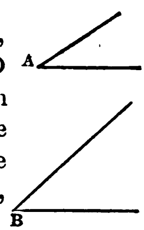


SOLUTION. Through C draw any line, CD, cutting AB in D. At C, by preceding problem, construct the angle ECD equal to CDB.* Then EC is parallel to AB, since the alternate interior angles are equal (I., 11).

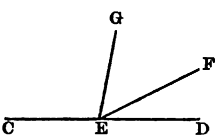
XXVI.

Problem. *Two angles of a triangle being given to find the third.*

Let A and B be the given angles.



SOLUTION. Draw a line, CD, and at a point, E, construct an angle FED = A, and GEF = B: then GEC is the third angle required, since it is the supplement of A + B (I., 4, Cor. 2, and I., 17, Cor. 1).

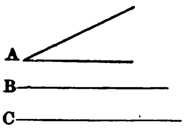


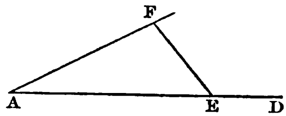
XXVII.

Problem. *Two sides and the included angle of a triangle being given to construct the triangle.*

Let B and C be the two given sides, and A the included angle.

* The student should give in full the entire construction of any problem referred to.

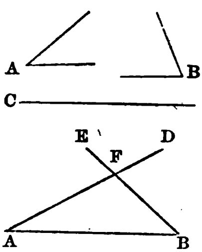




Solution. Draw the indefinite straight line AD, and at A construct the angle FAE = A. Take AE = B, AF = C, and join FE: then AEF is the triangle required; for it is constructed with the given lines and angle.

XXVIII.

Problem. *One side and two angles of a triangle being given to construct the triangle.*



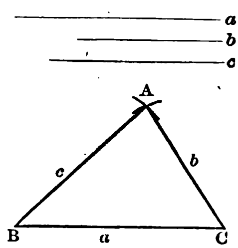
If one of the given angles is opposite the given side, find the other adjacent angle (26). We shall, therefore, always have a side and two adjacent angles given to construct the triangle. Let C be the given side, A and B, the adjacent angles of the required triangle.

Solution. Draw a line AB = C. At the extremities, A and B, construct angles equal to A and B respectively. The sides AD and BE intersecting in F, the triangle ABF will be constructed with the given data.

Scholium. If $\angle A + B =$ or $> 2R$, the problem is not possible; for the lines AD and BE will not meet (I, 17).

XXIX.

Problem. *The three sides of a triangle being given to construct the triangle.*



Let a, b, and c be the three given sides of the required triangle.

Solution. Draw BC = a. From B and C as centres, with radii respectively equal to c and b, describe arcs cutting each other in the point A.

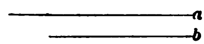
Draw BA and CA: then ABC will be the triangle required.

Scholium. The problem is impossible when any side, $a =$ or $> b + c$ (I., 16).

XXX.

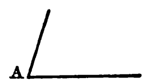
Problem. *Two sides of a triangle, and the angle opposite one of them, being given to construct the triangle.*

Let A be the given angle, a the opposite, and b an adjacent side.



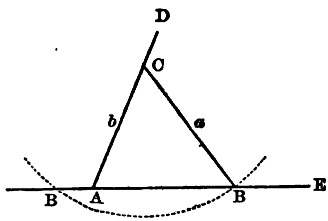
There may be three cases.

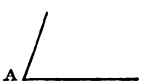
Case 1. $a > b$: then $\angle$ A may be an acute, right, or obtuse angle (I., 27).

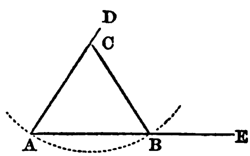


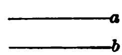
Solution. Construct an angle DAE = A. In one of the sides, AD, take AC $= b$.

From C as a centre, with a radius equal to a, draw an arc cutting the other side, AE, in two points, B and B′, one of which, B′, will be in AE extended, since $a > b$. Draw CB: then ACB will be the required triangle. In this case there will always be one, and only one solution.









Case 2. $a = b$: then $\angle$ A must be acute; and the triangle will be isosceles.

Solution. Construct as in Case 1.

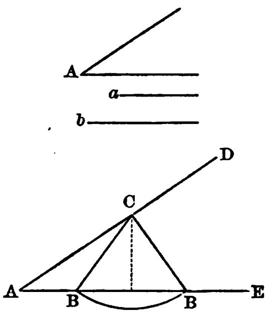
The cutting point, B′, will fall on A; and but one solution will be possible.

Case 3. $a < b$. $\angle$ A must be acute, since it lies opposite the less side, a.

Solution. Construct as in Case 1.

The two cutting points, B and B′, lie upon the same side

of A: then ACB or ACB′ is the required triangle; and the problem has two solutions.



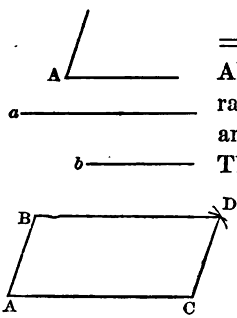
If, however, the side a be gradually diminished, the points B and B′ will approach each other, and fall together when b is equal to the perpendicular from C on AE; and there will be but one solution.

If a be less than this perpendicular, the problem is impossible.

XXXI.

Problem. *The adjacent sides of a parallelogram, and their included angle, being given to construct the parallelogram.*

Let A be the given angle, the lines a and b the adjacent sides.



SOLUTION 1. Construct an angle BAC = A. In the sides, lay off AC = a, and AB = b. From B and C as centres, with radii equal to a and b respectively, draw arcs intersecting in D. Draw BD and CD. The opposite sides are equal, and the figure is the parallelogram required (I., 38).

SOLUTION 2. Through B and C draw BD || AC, and CD || AB. They will intersect in a point, D; and ABDC is the parallelogram required.

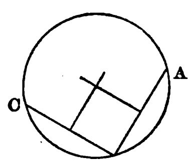
XXXII.

Problem. *To describe a circumference that shall pass through three given points not in the same straight line.*

Problem. *To find the centre of a given circumference.*

Problem. *To circumscribe a circle about a given triangle.*

The solution of these problems follows easily from 4.

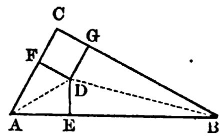


XXXIII.

Problem. *To inscribe a circle in a given triangle.*

Let ABC be the given triangle.

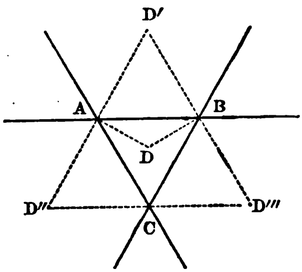
Solution. Bisect any two angles, A and B, by lines meeting in D.



From D draw DE, DF, and DG perpendicular to the sides of the triangle.

Then DE = DF = DG (I., 34): hence the circle described from D as a centre, with a radius equal to DE, will touch the three sides of the triangle in the points E, F, and G (11, Cor. 7), and will be inscribed in the triangle.

Scholium. If $\angle A + B > 2R$, that is, if AC and BC diverge, it is evident that a circle may be drawn in the same manner tangent to the three lines; and generally, if three straight lines intersect each other, four circles may be drawn tangent to them; also each intersecting point, A, is in the same straight line with the centres, two and two; thus, D′AD″ and ADD‴ are straight lines. (I., 32.)



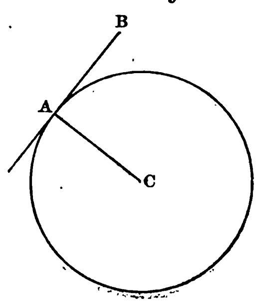
XXXIV.

Problem. *Through a given point to draw a tangent to a given circle.*

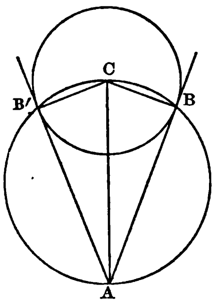
There may be two cases.

Case 1. The given point, A, lies on the circumference.

Solution. Draw the radius CA, and at A draw AB ⊥ CA: then AB will be the tangent required (11, Cor. 7).



Case 2. The given point, A, lies without the circle whose centre is C.

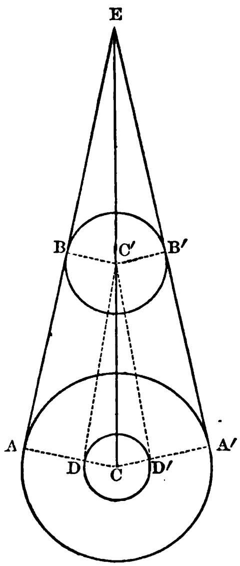


Solution. Upon AC as a diameter describe a circumference intersecting in the points B and B′. Draw AB and AB′. They will be tangent to the given circle, C.

Proof. Draw CB and CB′ radii of the given circle; then CBA and CB′A are right angles, being inscribed in semi-circles (11, Cor. 3): hence AB and AB′ are tangent to the circle C (11, Cor. 7).

Scholium. From a point without a circle, two tangents may always be drawn to the circumference, and they will be equal; for AB = AB′, since the right-angled triangles ABC and AB′C are equal (I., 29, Cor.).

XXXV.



Problem. *To draw a common tangent to two given circles.*

Let C and C′ be the centres of the two given circles, and C > C′.

1st, To draw an *exterior* common tangent.

Solution. From C as a centre, with a radius, CD, equal to the difference of the radii of the two given circles, describe the circumference DD′. From the point C′ draw a tangent, C′D, by the last problem. Draw CDA and C′B || CA. Draw BA: it will be the common tangent required.

Proof. ABC′D is a rectangle, since AD = || BC′ and ∠ ADC′ = R (I., 39, and I., 37, Cor. 4).

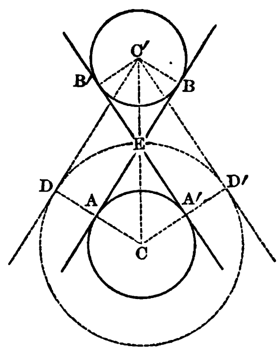
Hence AB is perpendicular to the radii at their extremities, and is tangent to the circumferences.

Upon the other side of CC′, a second exterior common tangent, A′B′, may be constructed, which will meet AB in CC′ produced; and the problem has two solutions.

If the circles are equal, draw the radii CA and C′B perpendicular to CC′.

AB will be the common tangent required.

2d, To draw an *interior* common tangent.



SOLUTION. From the centre C, with a radius equal to the sum of the radii of the given circles, describe the circumference DD′. From C′ draw the tangent C′D.

Draw CD and C′B ∥ CD. ABC′D is a rectangle, since AD $=\parallel$ C′B, and $\angle$ ADC′ $=$ R: hence AB is the interior common tangent required.

There will be two interior common tangents, AB and A′B′, which intersect each other in a point, E, of the line CC′.

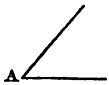
SCHOLIUM. If the two circles touch each other externally, the two interior common tangents fall together. If the circles cut each other, only the exterior tangents can be drawn. If the circles touch each other internally, the two exterior tangents fall together. If the circles are wholly within each other, there is no solution.

XXXVI.

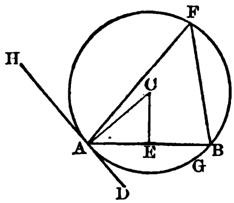
Problem. *Upon a given line to construct a segment that shall contain a given angle.*

Let AB be the given line, and A the given angle.

SOLUTION. At the point A construct the angle BAD = A.



Draw AC ⊥ AD, and at E, the middle point of AB, draw EC ⊥ AB, intersecting AC in C. From C as a centre, with a radius CA, describe the circle ABF: then AFB is the segment required.



PROOF. AD is a tangent, since it is perpendicular to the radius CA (11, Cor. 7): hence any angle in the segment AFB = BAD = A; for they have the same numerical measure, $\frac{1}{2}$ arc AGB (11, Cor. 10).

If the given angle be obtuse, construct BAH = A: then AGB is the required segment.

EXERCISE 1. The greatest and the least distances of any point from a circumference is measured on the diameter passing through that point.

EXERCISE 2. If from a point in a circumference chords be drawn, the locus of their middle points is a circumference.

EXERCISE 3. In any inscribed polygon, the sum of the 1st, 3d, 5th, &c., angles, is equal to the sum of the 2d, 4th, 6th, &c., angles.

EXERCISE 4. In any circumscribed polygon, the sum of the 1st, 3d, 5th, &c., sides, is equal to the sum of the remaining sides.

EXERCISE 5. If two circles cut each other, and diameters are drawn through one of the cutting points, the extremities of those diameters are in the same straight line with the other cutting point.

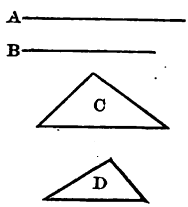
EXERCISE 6. The lines bisecting the angles contained by the opposite sides (produced) of an inscribed trapezium intersect at right angles.

BOOK III.

RATIO AND PROPORTION.

I.

1. Four magnitudes, A and B of one kind, and C and D of the same or another kind, are **proportional**, when the ratio of A to B is equal to that of C to D; that is, when $\frac{A}{B} = \frac{C}{D}$ (II., 7). Thus A and B may be two lines, C and D, two triangles. The proportion is often written A : B = C : D, and read, A is to B as C is to D.



2. The first and fourth terms, A and D, are called the *extremes;* the second and third, B and C, the *means:* the first and third, A and C, the *antecedents;* the second and fourth, B and D, the *consequents;* and the fourth term, D, a *fourth proportional* to the other three. Three quantities, A, B, and C, are proportional, when A : B = B : C. B is called a *mean proportional* between A and C.

3. Since the form A : B = C : D is equivalent to $\frac{A}{B} = \frac{C}{D}$, the following transformations may be made by simple algebraic operations. Multiplying both members by BD, we have A.D = B.C.

Hence *the product of the extremes is equal to the product of the means.*

If $A : B = B : C$, we have $B^2 = A.C$, or, $B = \sqrt{A.C}$; that is, *the mean proportional between two quantities is equal to the square root of their product.*

EXERCISE 1. Find the mean proportional between 6 and 24; between 5 and 10.

SCHOLIUM. It is important for the student to remember, that, in these transformations, the terms A, B, C, and D, are the numerical representatives of the four magnitudes; that is, A and B are the numbers of times a given unit is contained in the first two; C and D, the numbers of times the same or a different unit is contained in the third and fourth respectively.

4. Dividing $A.D = B.C$ successively by $B.D$, $C.D$, and $A.C$, we have,

$$\frac{A}{B} = \frac{C}{D}, \text{ or, } A : B = C : D \text{ I.},$$

$$\frac{A}{C} = \frac{B}{D}, \text{ or, } A : C = B : D \text{ II.},$$

$$\frac{D}{C} = \frac{B}{A}, \text{ or, } B : A = D : C \text{ III.}$$

Hence, *if the product of two quantities is equal to the product of two other quantities, one pair of factors may be made the extremes, and the other pair the means, of a proportion.*

EXERCISE 2. From $A.D = B.C$, write eight proportions.

5. Since the proportions II. and III. are true when I. is true, it follows, that, *if four quantities are proportional, the first is to the third as the second is to the fourth;* that is, they are proportional by **alternation.** Also *the second is to the first as the fourth to the third;* that is, they are proportional by **inversion.**

Thus if $A : B = C : D$,
then $A : C = B : D$ (by alternation),
and $B : A = D : C$ (by inversion).

6. By squaring, cubing, &c., and extracting roots of the equation $\frac{A}{B} = \frac{C}{D}$, or, $A : B = C : D$, we obtain the proportions, $A^2 : B^2 = C^2 : D^2$, $A^3 : B^3 = C^3 : D^3$, $\sqrt{A} : \sqrt{B} = \sqrt{C} : \sqrt{D}$, &c.; that is, *if four quantities are proportional, their like powers or roots are also proportional.*

7. Since $\frac{A}{B} = \frac{mA}{mB}$, it follows that $A : B = mA : mB$, where m may be a whole number or fraction; that is, *equimultiples, or like parts, of two quantities, have the same ratio as the quantities themselves.*

8. Given $\frac{A}{B} = \frac{C}{D}$, or, $A : B = C : D$.

Adding unity to both members of the equation, we have,

$$\frac{A+B}{B} = \frac{C+D}{D}, \text{ or, } A+B : B = C+D : D;$$

dividing this by the given equation, we have,

$$\frac{A+B}{A} = \frac{C+D}{C}, \text{ or, } A+B : A = C+D : C.$$

Hence, *if four quantities are in proportion, they are in proportion by* **composition.**

9. Subtracting unity from both members of $\frac{A}{B} = \frac{C}{D}$, we obtain, in like manner,

$$\frac{A-B}{B} = \frac{C-D}{D}, \text{ or, } A-B : B = C-D : D,$$

and $$\frac{A-B}{A} = \frac{C-D}{C}, \text{ or, } A-B : A = C-D : C.$$

Hence, *if four quantities are in proportion, they are in proportion by* **division.**

10. Dividing the first equation in (8) by the first in (9) we obtain

$$\frac{A+B}{A-B} = \frac{C+D}{C-D}, \text{ or } A+B : A-B = C+D : C-D.$$

Hence, *if four quantities are in proportion, they are in proportion by composition and division.*

11. If r equals the common ratio in the continued proportion, $A : B = C : D = E : F = G : H = \&c.$;

then $$r = \frac{A}{B} = \frac{C}{D} = \frac{E}{F} = \frac{G}{H} = \&c.,$$

and $Br = A$, $Dr = C$, $Fr = E$, $Hr = G$, &c.:

adding these equations, we have,

$$(B + D + F + H + \&c.)r = A + C + E + G + \&c.,$$

or, $$r = \frac{A + C + E + G + \&c.}{B + D + F + H + \&c.} = \frac{A}{B}.$$

Hence, *in a continued proportion, the sum of the antecedents is to the sum of consequents as any antecedent is to its consequent.*

12. If $$\frac{A}{B} = \frac{C}{D}, \text{ or, } A : B = C : D,$$

and $$\frac{E}{F} = \frac{G}{H}, \text{ or, } E : F = G : H,$$

we have, by multiplying these equations together,

$$\frac{A \times E}{B \times F} = \frac{C \times G}{D \times H}, \text{ or, } A \times E : B \times F = C \times G : D \times H.$$

Hence, *if the corresponding terms of two proportions be multiplied together, the products are proportional.*

13. If $$\frac{A}{B} = \frac{C}{D}, \text{ or, } A : B = C : D,$$

and $$\frac{B}{F} = \frac{G}{H}, \text{ or, } B : F = G : H;$$

then $$\frac{A}{F} = \frac{C \times G}{D \times H}, \text{ or } A : F = C \times G : D \times H.$$

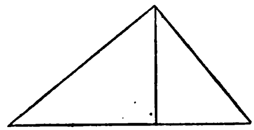
EXERCISE 3. If $A : C = B : D$, and $A : E = B : F$,
prove that $C : E = D : F$.

EXERCISE 4. If $A : C = B : D$, and $E : C = F : D$,
prove that $A : E = B : F$.

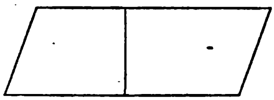
EXERCISE 5. If $A : B = B : C$,
prove that $A : C = A^2 : B^2$.

DEFINITIONS.

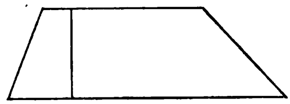
14. The *altitude of a triangle* is the perpendicular distance from the vertex of either angle to the opposite side, or the opposite side produced.



This angle is called the *vertical angle;* and the opposite side, the *base.*



15. The *altitude of a parallelogram* is the perpendicular distance between two opposite sides, which are called *bases.*



16. The *altitude of a trapezoid* is the perpendicular distance between its parallel sides.

17. The *area of a surface* is expressed by its ratio to a *unit of surface.* The unit generally adopted is a square whose side is a linear unit; as, one inch, one foot, &c.

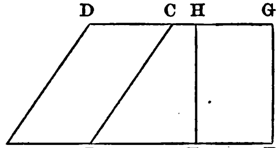
AREAS.

II.

Theorem. *The areas of parallelograms which have equal bases and equal altitudes are equal.*

Hypoth. The parallelograms ABCD and EFGH have equal altitudes, and the base AB = EF.

To be Proved. Area ABCD = EFGH.



Proof. Since the parallelograms have the same altitude, they may be placed between the same parallels, AF and DG: then, if the trapezoid AEHD be moved to the right, the side EH will fall on FG at the same time that AD falls on BC; since AB = EF.

Hence AEHD $\cong$ BFGC.

Subtracting BEHC from each member, we have

$$AEHD - BEHC = BFGC - BEHC,$$

or, $$ABCD = EFGH.$$

Cor. 1. The area of any parallelogram is equal to that of a rectangle which has the same base and altitude.

Cor. 2. It was shown (I., 37) that a diagonal divides a parallelogram into two congruent triangles: hence *the area of a triangle is one-half the area of a paralellogram which has an equal base and equal altitude.*

Cor. 3. The areas of triangles which have equal bases and equal altitudes are equal; for they are halves of equal parallelograms.

III.

Theorem. Conversely, *parallelograms, and also triangles, which have equal areas,*

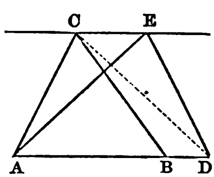
1st, *Have equal altitudes when their bases are equal.*

2d, *Have equal bases when their altitudes are equal.* (Demonstration indirect.)

IV.

Theorem. *If two triangles (or parallelograms) have equal altitudes, the ratio of their areas is equal to that of their bases.*

Hypoth. The triangles ACB and AED have equal altitudes.



To be Proved. $\dfrac{\triangle ACB}{\triangle AED} = \dfrac{\text{base } AB}{\text{base } AD}$.

Proof. Draw CD. Then ACD = AED (2, Cor. 3). Let the bases have a common measure, which is contained, for example, 9 times in AD, and 7 times in AB; then $\dfrac{AB}{AD} = \dfrac{7}{9}$.

Conceive lines drawn from the vertex, C, to the several points of division of the bases. ACD will be divided into

9 equal triangles (2, Cor. 3), of which ACB will contain 7: consequently $\frac{ACB}{ACD} = \frac{ACB}{AED} = \frac{7}{9}$:

hence $\frac{ACB}{AED} = \frac{AB}{AD}$.

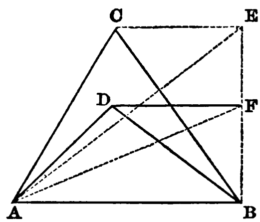
When the bases are incommensurable, the demonstration is the same as in II., 9.

Since the triangles ACB and AED are halves of parallelograms with the same bases and altitude, it follows that the same is true of parallelograms; for $\frac{2ACB}{2AED} = \frac{AB}{AD}$.

V.

Theorem. *If two triangles (or parallelograms) have equal bases, the ratio of their areas is equal to that of their altitudes.*

HYPOTH. The triangles ABD and ABC have the same base, AB, and the altitudes, BE and BF.



TO BE PROVED. $\frac{ABC}{ABD} = \frac{BE}{BF}$.

PROOF. Let BE $\perp$ AB. Draw DF and CE $\parallel$ AB: then BE and BF equal the altitudes of the given triangles; and we have
$\triangle$ ABC = ABE, also ABD = ABF (2, Cor. 3).

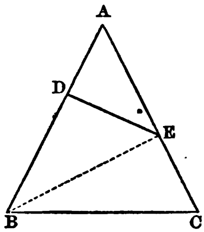
But $\frac{ABE}{ABF} = \frac{BE}{BF}$ (4): hence $\frac{ABC}{ABD} = \frac{BE}{BF}$.

The same is true of parallelograms.

VI.

Theorem. *If two triangles have an angle of the one equal to an angle of the other, the ratio of their areas is equal to that of the products of the sides which contain those angles.*

HYPOTH. The triangles ABC and ADE have an angle, A, in each equal.



TO BE PROVED. $\frac{ABC}{ADE} = \frac{AB \times AC}{AD \times AE}$.

PROOF. Let the triangles be so applied to each other, that the equal angles, A, shall coincide. Draw BE. The triangles ADE and ABE, having a common vertex, E, and their bases, AD and AB in the same straight line, have the same altitude: hence

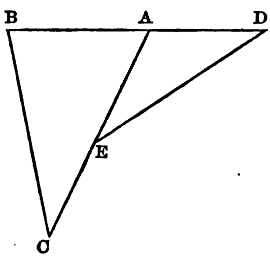
$$\frac{ABE}{ADE} = \frac{AB}{AD} \ (4).$$

Also the triangles ABC and ABE, having the same vertex, B, and their bases, AC and AE in the same straight line, have the same altitude: hence

$$\frac{ABC}{ABE} = \frac{AC}{AE} \ (4).$$

Multiplying these equations, and cancelling ABE, we have,

$$\frac{ABC}{ADE} = \frac{AB \times AC}{AD \times AE}.$$



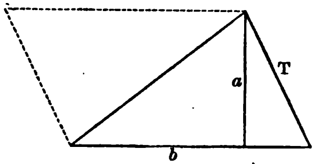
EXERCISE. If two triangles have an angle of the one that is a supplement to an angle of the other, the ratio of their areas is equal to that of the products of the sides which contain those angles; or,

$$\frac{ABC}{ADE} = \frac{AB \times AC}{AD \times AE}.$$

VII.

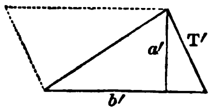
Theorem. *The areas of any two triangles (or parallelograms) are to each other as the products of their bases by their altitudes.*

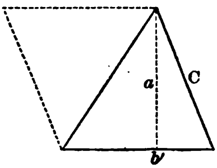
HYPOTH. Let T and T′ be two triangles, a and a' their altitudes, b and b' their bases.



TO BE PROVED. $\dfrac{T}{T'} = \dfrac{a \times b}{a' \times b'}$.

PROOF. Let C be a third triangle, whose altitude is a, and base, b'. Then (by 4 and 5) we have $\dfrac{T}{C} = \dfrac{b}{b'}$, and $\dfrac{C}{T'} = \dfrac{a}{a'}$.



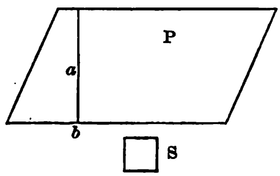


Multiplying these ratios, we obtain

$$\frac{T}{T'} = \frac{a \times b}{a' \times b'}.$$

The demonstration is the same if T and T′ represent parallelograms.

COR. Let P be any parallelogram, a its altitude, b its base, and S the square whose side is a linear unit:



then $\dfrac{P}{S} = \dfrac{a \times b}{1 \times 1} = a \times b.$

But, since S is the unit of surface, $\dfrac{P}{S}$ is the area of the parallelogram (Def. 17): hence,

Area of a parallelogram = base $\times$ altitude; also

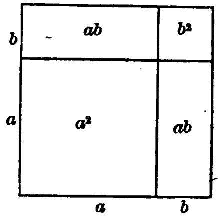
Area of a rectangle = length $\times$ breadth,

Area of a square = $(\text{side})^2$,

Area of a triangle = $\frac{1}{2}$ base $\times$ altitude.

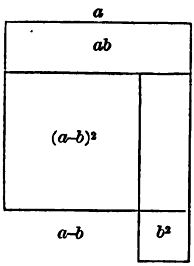
VIII.

Theorem 1. *If a and b are two lines, the square described on their sum is equal to the square on the first, a, plus twice the rectangle formed with the two lines, a and b, plus the square of the second, b; that is,*



$$(a+b)^2 = a^2 + 2ab + b^2.$$

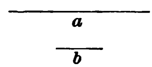
Cor. If $a = b$, the square on $(a+b)$ $= 4a^2$; that is, *the square on a line is equal to four times the square on half the line.*



Theorem 2. *If a and b are two lines, the square described on their difference is equal to the square on the first, a, minus twice the rectangle formed with the two lines, a and b, plus the square of the second, b; that is,*

$$(a-b)^2 = a^2 - 2ab + b^2.$$

Theorem 3. *The rectangle formed with the sum $(a+b)$ and difference $(a-b)$ of two lines, a and b, is equal to the difference of the squares on those lines; that is,*



$$(a+b)\ (a-b) = a^2 - b^2.$$

Construct the figures.

The truth of these theorems is evident from the figures, and they agree with the results found in algebra.

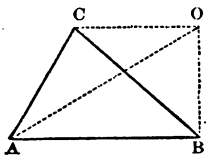
IX.

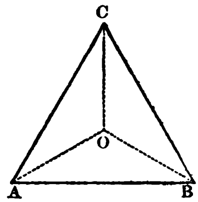
Definition. The perimeter of a polygon may be given in two ways. The expression ABCD . . . gives the direction from A to B to C . . . ; and ADCB gives the opposite direction. If ABCD denotes a positive area, ADCB may designate a negative area, and ABCD $= -$ ADCB.

Theorem. *If from any point,* O, *in the plane of a triangle,* ABC, *lines be drawn to the vertices of the angles, the algebraic*

sum of the triangles OAB, OBC, *and* OCA, *will equal the triangle* ABC.

PROOF. 1. If the point O lies within the triangle ABC, the triangles OAB, OBC, and OCA, are all positive, and together equal to ABC.





2. If O lies without the triangle, and between the sides AB and AC produced, the triangle OBC is negative, and

$$OAB + OBC + OCA = OAB - OCB + OCA = ABC.$$

3. If O lies between the sides of an angle vertical to an angle, A, of the triangle, it is evident that OAB and OCA are negative, while OBC is positive: hence their algebraic sum equals ABC. (Construct the figure.)

These are all the possible positions of O in reference to a side, BC.

Construct the figures for the positions of O in reference to the side AC or AB, and give the equations.

X.

Theorem. *If from any two points,* O *and* P, *in the plane of a polygon,* ABCD . . . MN *lines be drawn to the vertices of the angles, then will the triangles* OAB + OBC + OCD + . . . OMN + ONA = PAB + PBC + PCD + . . . PMN + PNA *when the order of the letters express the algebraic values of the triangles.*

PROOF. Draw OP, and construct the figure.

Then

$$\begin{aligned}
PAB &= OAB + OBP + OPA \ (9),\\
PBC &= OBC + OCP + OPB,\\
PCD &= OCD + ODP + OPC,\\
&\dots\dots\dots\dots\\
PMN &= OMN + ONP + OPM,\\
PNA &= ONA + OAP + OPN.
\end{aligned}$$

If we add these equations, and remember that OBP $=$ $-$ OPB, &c, the last two terms in the second member of the equations will cancel.

Hence PAB $+$ PBC $+$ PCD $+$ . . . PMN $+$ PNA $=$ OAB $+$ OBC $+$ OCD $+$ . . . OMN $+$ ONA.

Cor. If the polygon ABCD . . . is convex, and the point O lies within it, then it is evident that OAB $+$ OBC $+$ OCD $+$ . . . $=$ ABCD . . . Hence

$$\text{PAB} + \text{PBC} + \text{PCD} + \ldots = \text{ABCD} \ldots$$

whether the point P lies within or without the polygon.

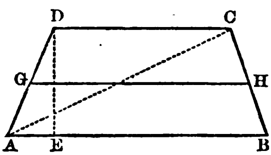
If ABCD . . . is a concave polygon, it may be divided into a number of convex polygons, and the same equation shown to be true.

XI.

Theorem. *The area of a trapezoid is equal to the product of its altitude by half the sum of its parallel sides.*

Hypoth. ABCD is a trapezoid, AB and CD are its parallel sides, and DE its altitude.

To be Proved. Area of ABCD $= \text{DE} \times \dfrac{\text{AB} + \text{CD}}{2}$.



Proof. Draw the diagonal AC. It divides the trapezoid into two triangles, whose bases are the parallel sides, and whose altitudes are DE.

Then $\triangle\,\text{ABC} = \text{DE} \times \dfrac{\text{AB}}{2}$ (7, Cor.),

and $\triangle\,\text{ADC} = \text{DE} \times \dfrac{\text{CD}}{2}$:

hence $\triangle\,\text{ABC} + \text{ADC} = \text{DE} \times \dfrac{\text{AB} + \text{CD}}{2}$;

or, $\text{ABCD} = \text{DE} \times \dfrac{\text{AB} + \text{CD}}{2}$.

Cor. Draw GH bisecting the non-parallel sides. Then $\text{GH} = \dfrac{\text{AB} + \text{CD}}{2}$ (I., 42, Cor. 3) : hence ABCD $=$ DE $\times$ GH ;

that is, *the area of a trapezoid is equal to the product of its altitude by a line joining the middle point of its non-parallel sides.*

EXERCISE. If AB = 12, DC = 8, DE = 4, what is the area of ABCD?

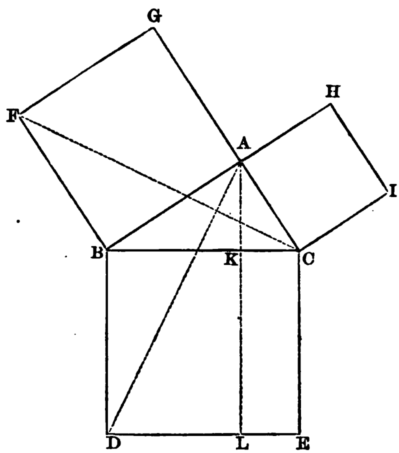
XII.

PYTHAGOREAN PROPOSITION.

Theorem. *The square described upon the hypothenuse of a right-angled triangle is equal to the sum of the squares described on the other two sides.*

HYPOTH. In the triangle ABC $\angle BAC = R$.

TO BE PROVED. $BC^2 = AB^2 + AC^2$.



PROOF. On the sides of the triangle construct the squares BE, BG, and CH. Join FC, AD, and draw AL || BD.

Then $\angle BAC + BAG = 2R$: hence CA and AG form one straight line (I., 6).

In the triangles ABD and FBC, we have,

AB = FB, being sides of a square,
BD = BC, " " "
and $\angle ABD = FBC$, each being $= R + ABC$:
hence $\triangle ABD \cong FBC$ (I., 20).
but $\triangle ABD = \frac{1}{2}BL$ (2, Cor. 2),
and $\triangle FBC = \frac{1}{2}BG$ (2, Cor. 2):
hence $BL = BG = AB^2$.

In like manner it may be shown, that

$$CL = CH = AC^2: \text{ hence}$$
$$BL + CL = AB^2 + AC^2; \text{ or,}$$
$$BC^2 = AB^2 + AC^2.$$

Cor. 1. *The square on one side of a right-angled triangle is equal to the square on the hypothenuse diminished by the square on the other side;* that is, $AB^2 = BC^2 - AC^2$.

Cor. 2. Since BL and KE have the same altitude, KL,

we have, $\quad BL : KE = BK : KC$ (4);

or, $\quad AB^2 : AC^2 = BK : KC$;

that is, *if the hypothenuse of a right-angled triangle be divided into segments by a perpendicular drawn from the vertex of the right angle, the squares on the sides about the right angle will be to each other as the adjacent segments.*

Cor. 3. In like manner we find,

$$BE : BL = BC : BK,$$

or, $\quad BC^2 : AB^2 = BC : BK$;

also $\quad BC^2 : AC^2 = BC : KC$;

that is, *the square on the hypothenuse is to the square on either side as the hypothenuse is to the segment adjacent to that side.*

Cor. 4. Let ABCD be a square, and AC a diagonal; then $AC^2 = AB^2 + BC^2 = 2AB^2$; that is, *the square described on the diagonal of a square is double the given square.*

Cor. 5. From $\quad AC^2 = 2AB^2$ (Cor. 4),

we have, $\quad AC^2 : AB^2 = 2 : 1$;

or, $\quad AC : AB = \sqrt{2} : 1$.

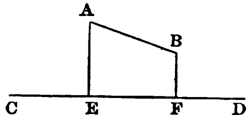
But $\sqrt{2}$ is an irrational quantity: hence *the diagonal and side of a square are incommensurable.*

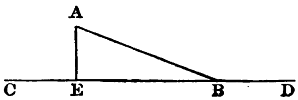
Scholium. By the expression BC^2 is meant the area of the square described on the line BC. If h, a, and b are the numerical representatives of the hypothenuse BC, and the sides AB and AC, the same relation exists between those numbers; that is, $h^2 = a^2 + b^2$; for h^2 is the measure of the square on BC, &c. (7, Cor.).

Exercise. If $AB = 4$, $AC = 3$, find BC, BK, and KC.

XIII.

DEF. The **projection** of a straight line, AB, upon a second line, CD, is the distance, EF, between the two perpendiculars let fall from the extremities, A and B, on CD. If one of the extremities, B, lies in CD, the distance BE is the projection of AB on CD.





XIV.

Theorem. *In any triangle, the square of a side opposite an* $\left[\begin{smallmatrix}\text{obtuse}\\ \text{acute}\end{smallmatrix}\right]$ *angle is equal to the sum of the squares of the other two sides* $\left[\begin{smallmatrix}\text{plus}\\ \text{minus}\end{smallmatrix}\right]$ *twice the rectangle of one of those sides, and the projection of the other upon it.*

HYPOTH. In the triangles ABC, the angle ACB is obtuse in Fig. 1, and acute in Figs. 2 and 3 (produced). CD is the projection of AC on CB (produced.)

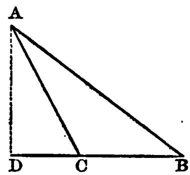


Fig. 1.

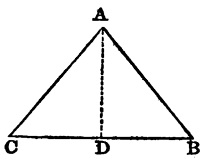


Fig. 2.

TO BE PROVED. $AB^2 = AC^2 + CB^2 \pm 2CB.CD$, in which $+$ applies to Fig. 1, and $-$ to Figs. 2 and 3.

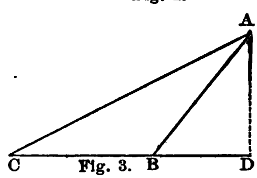


Fig. 3.

PROOF. $AB^2 = AD^2 + DB^2$ (12);

but $AD^2 = AC^2 - CD^2$,

and $DB^2 = (CB \pm CD)^2 = CB^2 + CD^2 \pm 2CB.CD$ (8).

Adding the last two equations, and substituting in the first, we have, $AB^2 = AC^2 + CB^2 \pm 2CB.CD$.

SCHOLIUM. The formula $AB^2 = AC^2 + CB^2 - 2CB.CD$ is applicable when C is an acute angle, a right angle, or an obtuse angle. For let the acute angle ACB be increased by turning AC about the point C, the projection CD will con-

stantly diminish, and be equal to zero when ACB is a right angle: hence the formula becomes,

$$AB^2 = AC^2 + CB^2, \text{ when } \angle ACB = R \text{ (see 12).}$$

Let the angle ACB be still increased, the projection CD will lie in the opposite direction, and be negative: hence the formula becomes,

$$AB^2 = AC^2 + CB^2 + 2CB.CD \text{ when } \angle ACB \text{ is obtuse.}$$

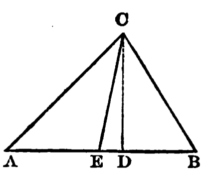
Cor. It is evident that $2CB.CD = 0$ only when $\angle ACB = R$: hence, if the square of one side of a triangle is equal to the sum of the squares of the other two sides, the angle between these sides is a right angle (converse of 12).

XV.

Theorem. *In any triangle, the sum of the squares described on two sides is equal to twice the square of half the third side, plus twice the square of the line drawn from the middle point of that side to the vertex of the opposite angle.*

Hypoth. In the triangle ABC, the line EC is drawn from E, the middle point of AB, to C, the vertex of the opposite angle.

To be Proved. $AC^2 + CB^2 = 2AE^2 + 2CE^2$.



Proof. From C draw the perpendicular CD to AB, or AB produced.

Then the obtuse-angled triangle, AEC, gives $AC^2 = AE^2 + CE^2 + 2AE.ED$ (14); and the acute-angled triangle, BEC, gives $CB^2 = BE^2 + CE^2 - 2BE.ED$ (14). Adding these equations, and remembering that $AE = BE$, we have, $AC^2 + CB^2 = 2AE^2 + 2CE^2$.

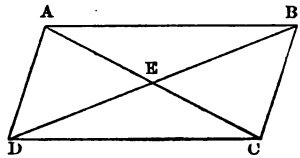
Exercise. If $AC = 8$, $BC = 6$, $AB = 7$, find CE.

XVI.

Theorem. *In any parallelogram, the squares of the sides are together equal to the squares of the diagonals.*

Hypoth. ABCD is a parallelogram, AC and BD are its diagonals.

To be Proved. $AB^2 + BC^2 + CD^2 + DA^2 = BD^2 + AC^2$.



Proof. The diagonals bisect each other in E (I., 41): hence, in the triangle ADC, we have,

$DA^2 + CD^2 = 2DE^2 + 2AE^2$;

and the triangle ABC gives,

$AB^2 + BC^2 = 2BE^2 + 2AE^2$.

Adding these equations, and remembering that BE = DE, and $4AE^2 = (2AE)^2 = AC^2$ (8, 1), we have,

$$AB^2 + BC^2 + CD^2 + DA^2 = BD^2 + AC^2.$$

Exercise. If AB = 12, BC = 6, AC = 14, find BD.

Def. If C be a point in the line AB, or in AB produced, AC and BC are called SEGMENTS of AB.

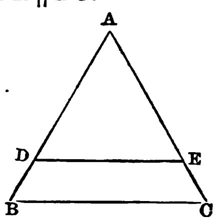
PROPORTIONAL LINES.

XVII.

Theorem. *A line drawn parallel to one side of a triangle, cutting the other two sides, divides them proportionally.*

Hypoth. In the triangle ABC, the line DE || BC.

To be Proved. AD : DB = AE : EC.



Proof. Let the lines AD and DB have a common measure, which is contained, for example, 7 times in AD, and 3 times in DB:

then AD : DB = 7 : 3.

Conceive, now, the line AB divided into 10 equal parts. D will be one point of division. Through the other points of division, suppose lines drawn parallel to BC, cutting AC. They will divide AE into 7 equal parts, and EC into 3 equal parts (I., 42).

Then AE : EC = 7 : 3:

hence AD : DB = AE : EC.

When the lines AD and DB are incommensurable, the demonstration is the same as in II., 9.

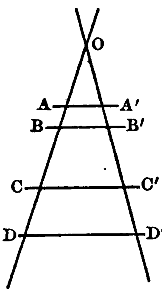
Exercise. If AD = 9, DB = 3, AE = 12, find EC.

Cor. 1. By composition, the proportion,

$$AD : DB = AE : EC, \text{ gives}$$
$$AB : DB = AC : EC, \text{ or}$$
$$AB : AD = AC : AE.$$

Cor. 2. If two lines are cut by any number of parallel lines, the corresponding parts intercepted are proportional.

Hypoth. The lines OD and OD′ are cut by the parallels, AA′, BB′, CC′, DD′, &c.



To be Proved. AB : A′B′ = BC : B′C′ = CD : C′D′.

Proof. 1st, Let the lines meet in a point, O.

Then OB : AB = OB′ : A′B′ (Cor. 1),

and OB : BC = OB′ : B′C′ :

hence OB : OB′ = AB : A′B′ (by alternation),

and OB : OB′ = BC : B′C′ " "

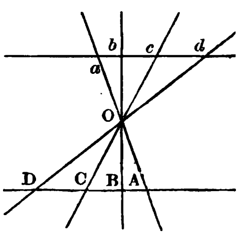
The last two proportions give

$$AB : A'B' = BC : B'C'.$$

2d, If the lines OD and OD′ are parallel, the part

$$AB = A'B', BC = B'C', CD = C'D' \text{ (I., 37 Cor. 1).}$$

and hence $AB : A'B' = BC : B'C' = CD : C'D'.$



Exercise. If two parallels intersect three or more lines that pass through a common point, their corresponding segments are proportional.

Hypoth. AD || *ad*.

To be Proved. OA : O*a* = OB : O*b* = OC : O*c* = OD : O*d*.

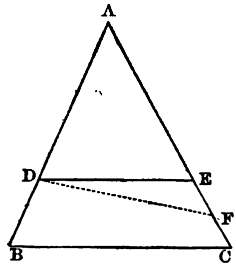
XVIII.

Theorem. Conversely, *if a straight line divides two sides of a triangle proportionally, it is parallel to the third side.*

Hypoth. In the triangle ABC, AD : DB = AE : EC.

To be Proved. The line DE || BC.

PROOF. If DE is not parallel to BC, suppose DF || BC. Then AD : DB = AF : FC (17); but AD : DB = AE : EC (Hypoth.): hence AF : FC = AE : EC, or AF : AE = FC : EC (alternation), which is impossible, since AF > AE, and FC < EC: hence DF is not parallel to BC.



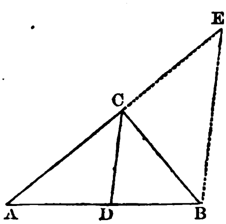
In the same manner it may be shown that any other line except DE is not parallel to BC.

Hence DE || BC.

XIX.

Theorem. *The line which bisects an angle of a triangle (or its exterior angle) divides the opposite side (or the opposite side produced) into segments that are proportional to the adjacent sides.*

HYPOTH. In the triangle ABC, the line CD bisects the angle ACB, dividing AB in D.

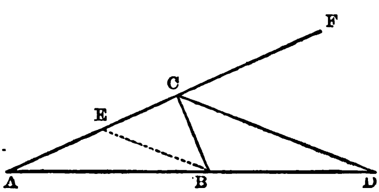


TO BE PROVED. AD : DB = AC : CB.

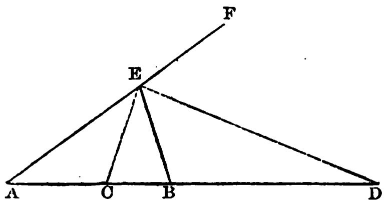
PROOF. In the line AC produced, take CE = CB, and join EB.

Then ∠ ACB = E + CBE (I., 18); but ∠ ACB = 2ACD (Hypoth.), and ∠ E + CBE = 2E (I., 25): hence ∠ ACD = E, and CD || EB (I., 11): therefore AD : DB = AC : CE or CB (17).

If CD bisects the exterior angle BCF, take CE = CB in the line AC; and the same demonstration gives, AD : DB = AC : CE or CB, remembering that FCB is the angle bisected, instead of ACB.



Exercise. If $AC = 14, AB = 15, BC = 7$, find AD and DB in each figure.



Cor. 1. If in a triangle, AEB, the line EC bisects the vertical angle AEB, and ED the exterior angle BEF, we have, by the proposition,

$$AC : CB = AE : EB,$$

and $$AD : DB = AE : EB:$$

hence $$AC : CB = AD : DB.$$

Exercise. If $AC = 6$, and $CB = 3$, find AD.

Def. When a straight line, AB, is divided by two points, C and D, such that $AC : CB = AD : DB$, the line AB is said to be divided **harmonically** in the points C and D.

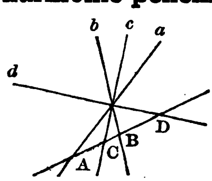
A and B, also C and D, are called *conjugate points.*

Cor. 2. When C lies near B, the line CB is small compared with AB, and DB must be small compared with AD: hence the points C and D approach B at the same time.

If $AC = CB$, it follows, from the proportion, that $AD = DB$, which is possible only when D is at an infinite distance. An isosceles triangle, AEB, shows this geometrically.

If $AC < CB$, then $AD < BD$; that is, the point D lies to the left of A, which is shown by the triangle AEB, when $AE < EB$.

Def. If the angles formed by two lines, a and b, be bisected by two other lines, c and d, the four lines form a **harmonic pencil.** (See Def. B. IX.)

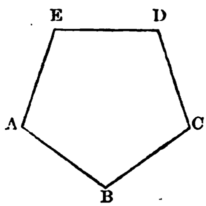


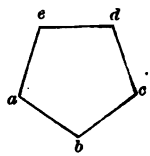
Cor. 3. Any line, AB, cutting a harmonic pencil, will be divided harmonically; and, if a line be drawn parallel to one line of a pencil, the two parts intercepted by the other three lines are equal. (See IX., 5 and 14 cor.)

SIMILAR POLYGONS.

XX.

DEF. Two polygons are **similar** when they are mutually equiangular, and have the sides about the equal angles proportional. The polygons ABCDE and *abcde* are similar when $\angle A = a$, $B = b$, $C = c$, &c., and $AB : ab = BC : bc = CD : cd =$ &c. Two equal angles, A and *a*, are called *homologous angles;* and two corresponding sides, AB and *ab*, are called *homologous* sides. Similarity is expressed by the symbol $\sim$; thus ABCDE $\sim$ *abcde* is read, ABCDE is similar to *abcde*.





XXI.

Theorem. *If, from any point in the side of a triangle, lines be drawn parallel to the other two sides, the two triangles thus formed will be similar to each other and to the given triangle.*

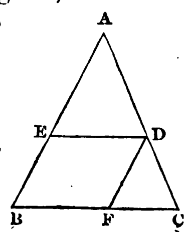
HYPOTH. In the triangle ABC, the line DE $\parallel$ BC, and DF $\parallel$ AB.

TO BE PROVED. $\triangle$ AED $\sim$ DFC $\sim$ ABC.

PROOF. The triangles are mutually equiangular, since their sides are parallel each to each (I., 12): also $AB : AE = AC : AD = BC : BF$ or ED (16); for ED $\parallel$ BC, and DF $\parallel$ AB:

hence $\triangle$ ABC $\sim$ AED.

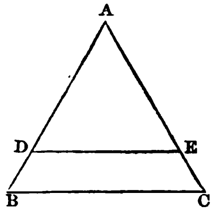
In like manner it may be shown that $\triangle$ ABC $\sim$ CFD.



XXII.

Theorem. *Two triangles are similar when they have two angles equal each to each.*

Hypoth. In the two triangles, ABC and ADE, $\angle A = A$, and $\angle B = D$.

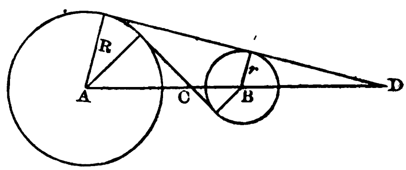


To be Proved. $\triangle ABC \sim ADE$.

Proof. Lay the triangle ADE upon ABC so that the equal angles A will coincide with each other, the point D will fall in AB, and E in AC.

Then, since $\angle B = D$, the side DE $\parallel$ BC (I., 11): hence $\triangle ABC \sim ADE$ (21).

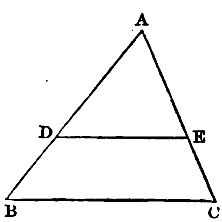
Exercise. Prove that the exterior and interior common tangents of two circles cut the line joining their centres harmonically (see II., 35).



XXIII.

Theorem. *Two triangles are similar when an angle of the one is equal to an angle of the other, and the sides including those angles are proportional.*

Hypoth. In the triangles ABC and ADE, $\angle A = A$, and $AB : AD = AC : AE$.



To be Proved. $\triangle ABC \sim ADE$.

Proof. Lay the triangle ADE upon ABC, so that the equal angles, A, will coincide with each other: the point D will fall in AB, and E in AC. Then, since

$$AB : AD = AC : AE \text{ (Hypoth.)},$$

the side DE $\parallel$ BC (18):

hence $\triangle ABC \sim ADE$ (21).

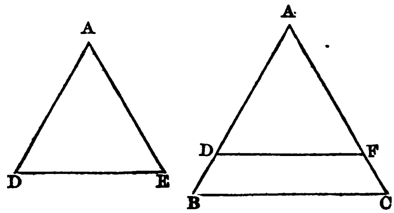
Exercise. Two triangles are similar when two sides of the one are proportional to two sides of the other, and the angles opposite the greater of these two sides equal.

XXIV.

Theorem. *Two triangles are similar when their corresponding sides are proportional.*

Hypoth. AB : AD = AC : AE = BC : DE.

To be Proved. $\angle$ A = A, $\angle$ B = D, $\angle$ E = C, that is, $\triangle$ ABC $\sim$ ADE.



Proof. In the side AB lay off the side AD, and through the point D draw DF $\parallel$ BC. Then AB : AD = AC : AF = BC : DF;

but AB : AD = AC : AE = BC : DE (Hypoth.).

These proportions give

AF = AE, and DF = DE;

then $\triangle$ ADF $\cong$ ADE (I., 24);

but $\triangle$ ADF $\sim$ ABC (21):

hence $\triangle$ ABC $\sim$ ADE.

XXV.

Theorem. *Two triangles are similar when their sides are parallel each to each, or perpendicular each to each.*

Hypoth. Two triangles, ABC and *abc*, have their sides parallel each to each, that is, AB $\parallel$ *ab*, AC $\parallel$ *ac*, and BC $\parallel$ *bc;* or perpendicular each to each, that is AB $\perp$ *ab*, AC $\perp$ *ac*, and BC $\perp$ *bc*.

To be Proved. $\triangle$ ABC $\sim$ *abc*.

Proof. Since the sides of the homologous angles are parallel each to each (I., 12), or perpendicular each to each (I., 35, Cor. 5), these angles are either equal or supplements of each other: hence three hypotheses may be made, —

1st, $A + a = 2R$, $B + b = 2R$, $C + c = 2R$;
2d, $\angle A = a$, $B + b = 2R$, $C + c = 2R$;
3d, $\angle A = a$, $B = b$, whence $\angle C = c$ (I., 17, Cor. 2).

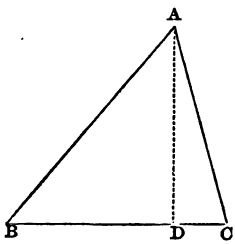
The 1st and 2d hypotheses make the angles of the two triangles together greater than 4R, which is impossible (I., 17): hence the 3d hypothesis is the only one that can be admitted; that is, the triangles must be mutually equiangular and similar (22).

SCHOLIUM. *Two parallel sides, or two perpendicular sides, in the two triangles, are homologous.*

EXERCISE. In similar triangles, homologous altitudes are to each other as any pair of homologous sides.

XXVI.

Theorem. *The areas of two similar triangles are to each other as the squares of two homologous sides or altitudes.*



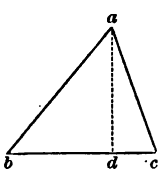
HYPOTH. $\triangle ABC \sim abc$.

TO BE PROVED. $ABC : abc = AB^2 : ab^2$.

PROOF. The angle $A = a$ (Hypoth.):

hence $$\frac{ABC}{abc} = \frac{AB \times AC}{ab \times ac} \quad (6).$$

But $$\frac{AB}{ab} = \frac{AC}{ac} \quad \text{(Hypoth.)}.$$



Multiplying each member of this equation by $\frac{AB}{ab}$, we have, $\frac{AB^2}{ab^2} = \frac{AB \times AC}{ab \times ac}$:

hence $$\frac{ABC}{abc} = \frac{AB^2}{ab^2};$$

or $$ABC : abc = AB^2 : ab^2.$$

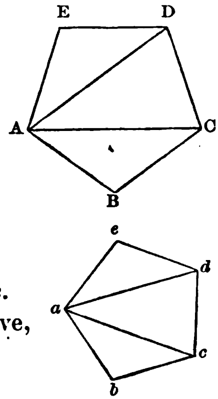
It is easy to show that the altitudes AD and ad are to each other as AB is to ab. Hence, also, $ABC : abc = AD^2 : ad^2$.

EXERCISE. If $BC = 10$, area $ABC = 60$, and $bc = 6$, find the area of abc.

XXVII.

Theorem. *Two similar polygons may be divided into the same number of triangles similar each to each.*

HYPOTH. ABCDE and *abcde* are similar polygons, divided into triangles by diagonals drawn from the vertices of two homologous angles, A and *a*.



TO BE PROVED. $\triangle$ ABC $\sim$ *abc*, ACD $\sim$ *acd*, &c.

PROOF. $\angle$ B = *b*, and AB : BC = *ab* : *bc* (Hypoth.) : hence

$$\triangle \text{ABC} \sim abc \ (23).$$

Also $\angle$ BCD = *bcd* (Hypoth.),
and $\angle$ ACB = *acb*, since $\triangle$ ABC $\sim$ *abc*.
Subtracting these two equations, we have,

$$\angle \text{ACD} = acd.$$

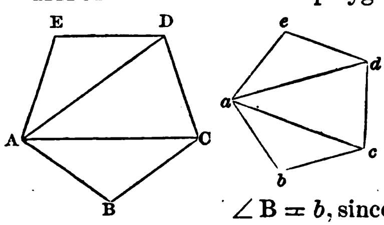
Again, BC : *bc* = CD : *cd* (Hypoth.),
and BC : *bc* = AC : *ac*, since $\triangle$ ABC $\sim$ *abc* :
hence AC : *ac* = CD : *cd*, and $\triangle$ ACD $\sim$ *acd* (23).

In the same manner, it may be shown that the remaining triangles are similar each to each.

XXVIII.

Theorem. Conversely, *if two polygons are composed of the same number of triangles similar each to each, and similarly situated, the polygons are similar.*

HYPOTH. In the two polygons, ABCDE and *abcde*, $\triangle$ ABC $\sim$ *abc*, ACD $\sim$ *acd*, ADE $\sim$ *ade*.



TO BE PROVED. ABCDE $\sim$ *abcde*.

PROOF. 1. The angles are equal each to each; for $\angle$ B = *b*, since $\triangle$ ABC $\sim$ *abc* (Hypoth.),

also $\angle$ ACB $=$ *acb*, since $\triangle$ ABC $\sim$ *abc*,
and $\angle$ ACD $=$ *acd*, since $\triangle$ ACD $\sim$ *acd* (Hypoth.).
Adding the last two equations, we have,

$\angle$ BCD $=$ *bcd*.

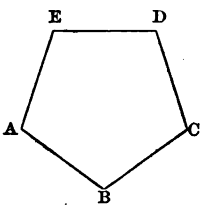
In the same manner, it may be shown that the remaining angles are equal each to each.

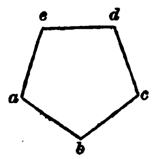
2. The homologous sides are proportional; for

AB : *ab* $=$ BC : *bc* $=$ AC : *ac*, since $\triangle$ ABC $\sim$ *abc*,
and AC : *ac* $=$ CD : *cd*, since $\triangle$ ACD $\sim$ *acd*:
hence AB : *ab* $=$ BC : *bc* $=$ CD : *cd*, &c.,
and the polygons are similar.

XXIX.

Theorem. *The perimeters of two similar polygons are to each other as any two homologous sides.*





Hypoth. ABCDE $\sim$ *abcde*.

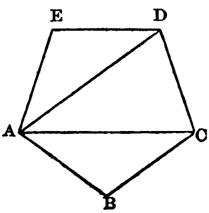
To be Proved. AB $+$ BC $+$ CD $+$ . . . : *ab* $+$ *bc* $+$ *cd* $+$ . . . $=$ AB : *ab*.

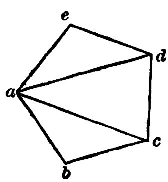
Proof. AB : *ab* $=$ BC : *bc* $=$ CD : *cd* $=$, &c. (Hypoth.): hence AB $+$ BC $+$ CD . . . : *ab* $+$ *bc* $+$ *cd* . . . $=$ AB : *ab* (1).

XXX.

Theorem. *The areas of two similar polygons are to each other as the squares of two homologous sides.*

Hypoth. ABCDE $\sim$ *abcde*.





To be Proved. ABCDE : *abcde* $=$ AB^2 : ab^2.

Proof. Let the polygons be divided into triangles similar each to each, by diagonals drawn from the vertices A and *a* (27).

Then $\triangle ABC : abc = AC^2 : ac^2$ (26),

and $\triangle ACD : acd = AC^2 : ac^2$:

hence $\triangle ABC : abc = ACD : acd$.

In the same manner, we may find

$$\triangle ACD : acd = ADE : ade:$$

hence $\triangle ABC : abc = ACD : acd = ADE : ade$,

and $\triangle ABC + ACD + ADE : abc + acd + ade = ABC : abc = AB^2 : ab^2$;

or $ABCDE : abcde = AB^2 : ab^2$.

EXERCISE. If AB = 6, $ab = 5$, area ABCDE = 100, find area $abcde$.

XXXI.

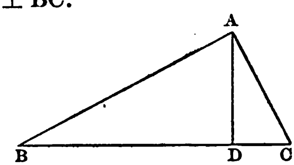
Theorem. *If, in a right-angled triangle, a perpendicular is drawn from the vertex of the right angle to the hypothenuse,*

1, *The triangles on each side of the perpendicular are similar to the given triangle, and to each other;*

2, *Each side is a mean proportional between the hypothenuse and its segment adjacent to that side;*

3, *The perpendicular is a mean proportional between the two segments of the hypothenuse.*

HYPOTH. $\angle BAC = R$, and $AD \perp BC$.



TO BE PROVED. 1. $\triangle ABC \sim DBA \sim DAC$.

PROOF. $\angle BAC = BDA = R$, and $\angle B = B$:
hence $\triangle ABC \sim DBA$ (22).
Also $\angle BAC = ADC$, and $C = C$:

hence $\triangle ABC \sim DAC$.

TO BE PROVED. 2. $BC : BA = BA : BD$,
and $BC : CA = CA : DC$.

These follow directly from

$$\triangle ABC \sim DBA,$$
and $\triangle ABC \sim DAC$.

To be Proved. 3. BD : AD = AD : DC.

This follows from $\triangle$ ABD $\sim$ CAD.

Cor. From the proportions,

$$BC : BA = BA : BD, \text{ and } BC : CA = CA : DC,$$

we have, $BA^2 = BC \times BD$, and $CA^2 = BC \times DC$:

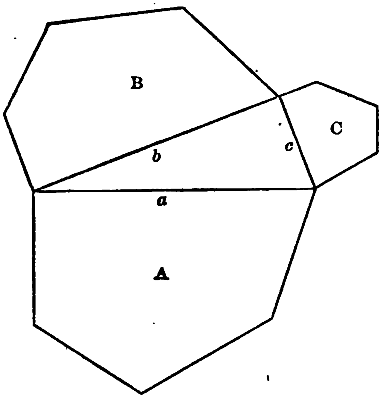
hence $BA^2 + AC^2 = BC(BD + DC)$;

or, $BA^2 + AC^2 = BC^2$, which was proved in 12.

Exercise. If BA = 4, AC = 3, find BD and AD.

XXXII.

Theorem. *Any polygon described upon the hypothenuse of a right-angled triangle is equal to the sum of two similar polygons described upon the other two sides, when the three sides of the triangle are homologous sides of the polygons.*



Hypoth. a is the hypothenuse, b and c the sides, of a right-angled triangle: A, B, and C are similar polygons, whose homologous sides are a, b, and c.

To be Proved. A = B + C.

Proof. $B : C = b^2 : c^2$ (30), and

$$B + C : C = b^2 + c^2 : c^2 \text{ (by composition).}$$

Also $A : C = a^2 : c^2$,

but $a^2 = b^2 + c^2$ (12):

hence $A = B + C$.

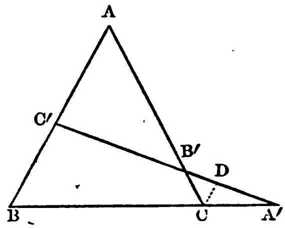
TRANSVERSALS.

Def. A straight line cutting any system of lines is called a **transversal**.

XXXIII.

Theorem. *If a transversal cuts the sides of a triangle (produced, if necessary), dividing them into six segments, the product of three segments whose extremities are not contiguous is equal to the product of the other three segments.*

HYPOTH. The transversal A′B′C′ cuts the triangle ABC.



TO BE PROVED. $AB' \times BC' \times CA' = AC' \times BA' \times CB'$.

PROOF. Draw CD ∥ AB; then

$\triangle CDB' \sim AC'B'$ (25):

hence $\dfrac{CD}{CB'} = \dfrac{AC'}{AB'}$.

Also $\triangle CA'D \sim BA'C'$:

hence $\dfrac{CA'}{CD} = \dfrac{BA'}{BC'}$.

Multiplying these equations together, we have,

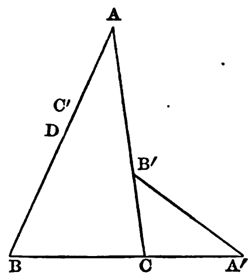
$$\frac{CA'}{CB'} = \frac{AC'}{AB'} \times \frac{BA'}{BC'};$$

clearing of fractions,

$$AB' \times BC' \times CA' = AC' \times BA' \times CB'.$$

XXXIV.

Theorem. Conversely, *if three points be taken on the sides of a triangle (one or all being on the sides produced), dividing them into six segments such that the product of three segments whose extremities are not contiguous is equal to the product of the other three segments, the three points will lie in the same straight line.*



HYPOTH. A′, B′, C′, are three points in the sides of the triangle ABC, so taken, that

$AB' \times BC' \times CA' = AC' \times BA' \times CB'$.

TO BE PROVED. A′, B′, C′, lie in the same straight line.

PROOF. Draw A′B,′ and produce it until it cuts the side AB in some point,

D; then $AB' \times BD \times CA' = AD \times BA' \times CB'$ (33);
but $AB' \times BC' \times CA' = AC' \times BA' \times CB'$ (Hypoth.).

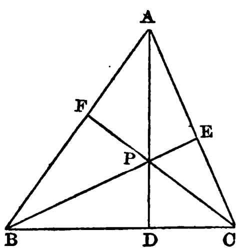
Dividing the first equation by the second, and cancelling, we have, $\frac{BD}{BC'} = \frac{AD}{AC'}$, which is evidently possible only when D and C′ coincide with each other. Hence A′, B′, C′, lie in the same straight line.

XXXV.

Theorem. *The lines drawn from the vertices of a triangle, through any point of its plane, divide the sides into segments such that the product of three segments whose extremities are not contiguous is equal to the product of the other three segments.*

Hypoth. From the vertices of a triangle, ABC, the lines AD, BE, and CF, are drawn through a point, P, which may be within or without the triangle.

To be Proved. $AF \times BD \times CE = BF \times AE \times CD$.



Proof. The triangle ABD, being cut by CF, gives $AF \times BC \times DP = BF \times CD \times AP$ (33).

Also ADC, being cut by BE, gives $BD \times CE \times AP = AE \times BC \times DP$ (33).

Multiplying these equations, member by member, and reducing, we have,

$AF \times BD \times CE = BF \times AE \times CD$.

XXXVI.

Theorem. Conversely, *if three points are taken on the sides of a triangle (no point or two being on the sides produced), so that the product of three segments whose extremities are not contiguous is equal to the product of the other three segments, then will the lines drawn from these points to the vertices of the opposite angles meet in one point.*

The demonstration is similar to that of 34.

COR. 1. The lines drawn from the vertices of a triangle to the middle of the opposite sides meet in one point.

For if D, E, and F are the centres of the sides of the triangle ABC, we have,

$$BD = DC,\ CE = EA,\ AF = FB:$$

hence $$BD \times CE \times AF = DC \times EA \times FB.$$

COR. 2. The bisectors of the angles of a triangle meet in one point.

For if AD, BE, and CF bisect the angles of the triangle ABC, we have, $\frac{BD}{DC} = \frac{BA}{AC}, \frac{CE}{EA} = \frac{CB}{BA}, \frac{AF}{FB} = \frac{AC}{CB}$ (19):

hence $$\frac{BD \times CE \times AF}{DC \times EA \times FB} = \frac{BA \times CB \times AC}{AC \times BA \times CB} = 1;$$

or, $$BD \times CE \times AF = DC \times EA \times FB.$$

COR. 3. The altitudes of a triangle meet in one point.

For if AD, BE, and CF are perpendicular to the sides of the triangle, we have,

$$\triangle ADC \sim BEC,\ ADB \sim CFB,\ BEA \sim CFA \ (22);$$

from which may readily be obtained the equation,

$$BD \times CE \times AF = DC \times EA \times FB.$$

COR. 4. If AD, BE, and CF have one point common, and D bisects the side BC, then the line EF is parallel to BC.

For $BD \times CE \times AF = DC \times EA \times FB$:

hence $CE \times AF = EA \times FB$, since $BD = DC$,

and $AF : FB = EA : CE$ (1);

that is, $EF \parallel BD$ (18).

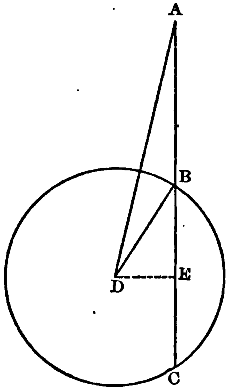
COR. 5. Conversely, if EF is parallel to BD, and the lines AD, BE, and CF, have one point common, it may be easily shown that $BD = DC$.

LINES ON THE CIRCLE.

XXXVII.

Theorem. *If through a point a line be drawn cutting the circumference of a circle, the product of the two segments is equal to the square of the distance of the point from the centre, minus the square of the radius.*

Hypoth. Let A be the point, and AC any line cutting the circumference whose centre is D.



To be Proved. $AC \times AB = AD^2 - BD^2$.

Proof. Draw $DE \perp BC$;

then $BE = EC$ (II., 4, Cor.),

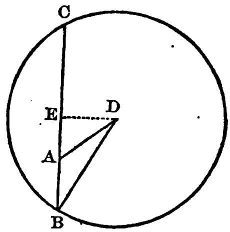
$AC = AE + EB$, and $AB = AE - EB$:

hence $AC \times AB = AE^2 - EB^2$;

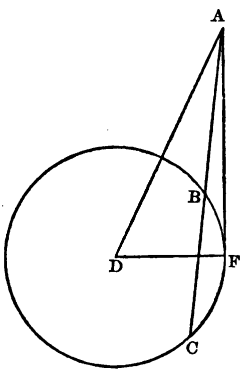
but $AE^2 = AD^2 - DE^2$ (12, Cor.),

and $EB^2 = BD^2 - DE^2$:

hence $AC \times AB = AD^2 - BD^2$.



Cor. 1. If the point A is without the circle, $AD > BD$; and hence $AD^2 - BD^2$, or its equal $AC \times AB$, is positive. Consequently AC and AB have the same sign; that is, they run in the same direction. If A lies within the circle, $AD < BD$, and hence $AC \times AB$ is negative; that is, they run in opposite directions.



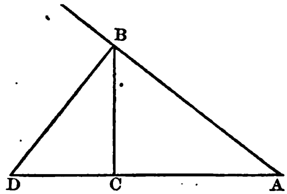
Cor. 2. If the line be a tangent, as AF, the two segments are equal; and $AF^2 = AD^2 - FD^2$, or $AF^2 = AC \times AB$, which gives also $AC : AF = AF : AB$.

The square of the tangent AF is called the **power of the point** A in reference to the circle. The power of a point within the circle is equal to the product of the two segments of a chord drawn through this point, and is negative: hence its square root is imaginary; that is, the tangent drawn from a point within a circle is an imaginary line.*

* An imaginary quantity is the indicated even root of a negative quantity; as, $\sqrt{-4}$.

COR. 3. If from any point without or within a circle, two or more lines be drawn cutting the circumference, the products of their segments are equal.

COR. 4. The power of a point, A, in reference to all the circles which pass through two points, B and C, is the same, $AB \times AC$; that is, the tangents drawn from this point to the circles are all equal.



EXERCISE. Given AD, AC, and $\angle A$. Find the point B, in the line AB, at which the angle DBC will have the greatest value.

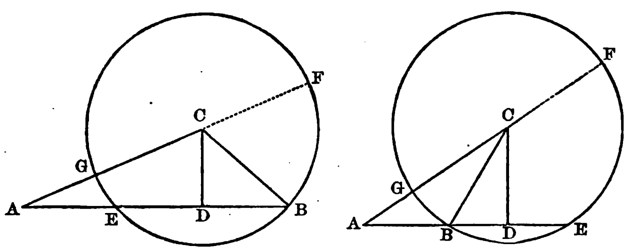
Ans. $AB^2 = AD \times AC$.

XXXVIII.

Theorem. *If, from the vertex of an angle of a triangle, a perpendicular be drawn to the opposite side, the product of the sum and difference of the segments of that side is equal to the product of the sum and difference of the other two sides.*

HYPOTH. In the triangle ABC, $CD \perp AB$.

TO BE PROVED. $(AD + DB)(AD - DB) = (AC + CB)(AC - CB)$.



PROOF. From C as a centre, with a radius CB, which is not greater than AC, draw the circle GBF. Produce AC to F.

Then $\quad AB \times AE = AF \times AG$ (37, Cor. 3);

but $\quad AB = AD \pm DB,\ AE = AD \mp DB,$

where the upper signs are taken when $\angle$ ABC is acute, and the lower when $\angle$ ABC is obtuse;

also $\quad AF = AC + CB,\ AG = AC - CB.$

Substituting these values, we have,

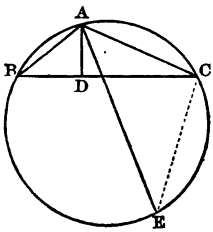
$$(AD + DB)\ (AD - DB) = (AC + CB)\ (AC - CB).$$

Exercise. If $AC = 10$, $CB = 6$, $AB = 12$, find AD.

XXXIX.

Theorem. *In any triangle, the product of two sides is equal to the product of the perpendicular let fall upon the third side from the vertex of the opposite angle, and the diameter of the circumscribed circle.*

Hypoth. In the triangle ABC, AD $\perp$ BC, and AE is the diameter of the circumscribed circle.



To be Proved. $AB \times AC = AD \times AE$.

Proof. Join EC. Then $\angle ADB = ACE = R$ (II., 11, Cor. 3);

also $\angle ABD = AEC$ (II., 11, Cor. 2):

hence $\quad \triangle ABD \sim AEC$ (22),

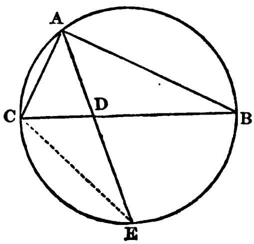
and $\quad AB : AE = AD : AC$,

or $\quad AB \times AC = AD \times AE$.

Exercise. If $AB = 8$, $AC = 12$, $AD = 6$, find diameter of the circumscribed circle.

XL.

Theorem. *In any triangle, the product of two sides is equal to the product of the segments of the third side made by a line bisecting the opposite angle, plus the square of that line.*



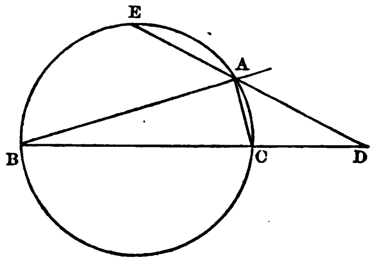
Hypoth. In $\triangle$ ABC, the line AD bisects $\angle$ BAC.

To be Proved. $AB \times AC = BD \times DC + AD^2$.

Proof. About ABC circumscribe a circle. Produce AD to E, and join EC.

Then $\angle B = E$ (II., 11, Cor. 2),
and $\angle BAD = EAC$ (Hypoth.):
hence $\triangle ABD \sim AEC$ (22),
and $AB : AE = AD : AC$,
which gives $AB \times AC = AE \times AD = (DE + AD)AD = DE \times AD + AD^2$;

but $DE \times AD = BD \times DC$ (37, Cor. 3):
hence $AB \times AC = BD \times DC + AD^2$.



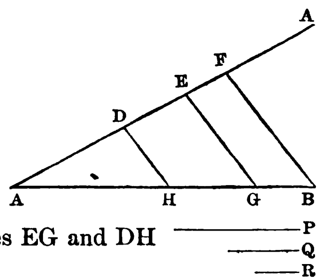
COR. If AD bisects the exterior angle of the triangle, the same demonstration gives, $AB \times AC = BD \times DC - AD^2$.

PROBLEMS.

XLI.

Problem. *To divide a given line into parts proportional to given lines.*

Let it be required to divide AB into parts proportional to P, Q, and R.



SOLUTION. From one extremity, A, draw the indefinite line AF; lay off $AD = P$, $DE = Q$, $EF = R$; draw FB; and through the points E and D draw the lines EG and DH parallel to FB.

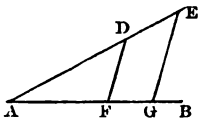
Then $AH : AD = HG : DE = GB : EF$ (17, Cor. 2),
or $AH : P = HG : Q = GB : R$;
that is, H and G are the points of division required.

COR. AB may be divided into any number of equal parts by taking the same number of equal parts on AF, and completing the construction.

XLII.

Problem. *To find a fourth proportional to three given lines.*

Let it be required to find a fourth proportional to P, Q, and R.



P

Q

R

SOLUTION. Draw AB and AE, making any angle with each other. Lay off AD = P, DE = Q, and AF = R. Join DF, and through the point E draw EG parallel to DF.

Then AD : DE = AF : FG (17),

or P : Q = R : FG;

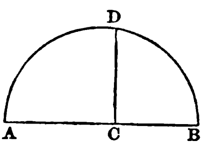
that is, FG is a fourth proportional to P, Q, and R.

COR. If AD = P, and DE = AF = Q, the line FG is a third proportional to P and Q; that is, P : Q = Q : FG.

XLIII.

Problem. *To find a mean proportional between two given lines.*

Let it be required to find a mean proportional between the lines P and Q.



P

Q

SOLUTION. On an indefinite line, AB, take AC = P, and CB = Q. On AB as a diameter describe a semi-circumference, and at C erect the perpendicular CD.

Then AC : CD = CD : CB (31, 3);

that is, CD is the mean proportional required.

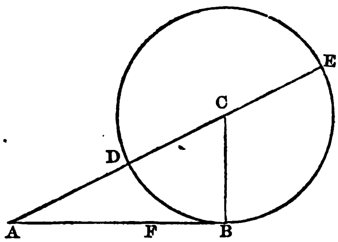
XLIV.

Problem. *To divide a line in* **extreme and mean ratio**; *that is, to divide a line into two parts, such that one of the parts shall be a mean proportional between the whole line and the other part.*

Let it be required to find a point, F, in the line AB, such that

AB : AF = AF : FB.

SOLUTION. Draw $BC \perp AB$, and take $BC = \frac{1}{2}AB$. With C as a centre, and a radius CB, draw the circumference DBE. Through A and C draw a straight line cutting the circumference in D and E. On AB lay off $AF = AD$. Then F is the point required; that is, $AB : AF = AF : FB$.



PROOF. Since AB is a tangent (II., 11, Cor. 8), and AE a secant, we have,

$$AE : AB = AB : AD \text{ or } AF \quad (37, \text{Cor. } 2),$$

by division, $AE - AB : AB = AB - AF : AF$;

but $AE - AB = AE - DE = AF$,

and $AB - AF = FB$:

hence $AF : AB = FB : AF$; or,

by inversion, $AB : AF = AF : FB$.

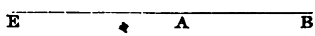
COR. By composition, the proportion

$$AE : AB = AB : AD$$

gives $AE + AB : AE = AB + AD : AB$,

or, $AE + AB : AE = AE : AB$;

that is, if the line AE be added to AB, the whole line EB will be divided by the point A in extreme and mean ratio.



By inversion, the last proportion becomes,

$$AB : AE = AE : AE + AB;$$

that is, the point E divides AB *externally* in extreme and mean ratio.

SCHOLIUM. Given AB; required to find AF and FB.

Let $x = AF$; then $FB = AB - x$,

and $AB : x = x : AB - x$,

which gives $x^2 = AB^2 - AB.x$,

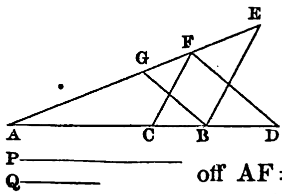
and $x = +\frac{AB}{2}(\sqrt{5} - 1)$; or, $x = -\frac{AB}{2}(\sqrt{5} + 1)$.

The second value of x gives AE when the point E divides the line AB externally.

XLV.

Problem. *To divide a line harmonically in a given ratio.*

Let AB be the given line, and $\frac{P}{Q}$ the given ratio. It is required to find two points, C and D, such that $\frac{AC}{CB} = \frac{AD}{DB} = \frac{P}{Q}$ (19, Def.).



Solution. From the point A draw an indefinite line, AE. Lay off AF = P, also FE and FG = Q. Draw BE, BG, FC || BE, and FD || BG.

Then $$\frac{AC}{CB} = \frac{AF}{FE} \text{ (17)},$$

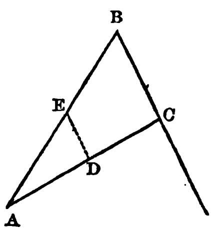
and $$\frac{AD}{DB} = \frac{AF}{FG} = \frac{AF}{FE} = \frac{P}{Q}:$$

hence $$\frac{AC}{CB} = \frac{AD}{DB} = \frac{P}{Q}.$$

XLVI.

Problem. *Through a given point in a given angle to draw a line so that the segments between the point and sides of the angle shall be equal.*

Let ABC be the given angle, and D the given point. It is required to draw a line, ADC, so that AD = DC.



Solution. Through D draw DE || BC, lay off EA = EB, and draw ADC. Then AD = DC; for

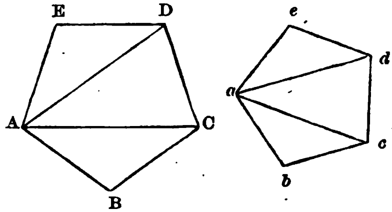
AE : EB = AD : DC (17);

but AE = EB: hence AD = DC.

XLVII.

Problem. *On a given straight line to construct a polygon similar to a given polygon.*

Let *ab* be the given straight line, and ABCDE the given polygon. It is required to construct a poylgon *abcde* $\sim$ ABCDE.



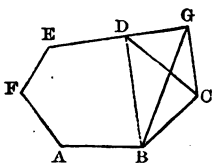
SOLUTION. From A draw the diagonals AC and AD. Construct $\angle$ *abc* = ABC, and $\angle$ *bac* = BAC (II., 24). Then $\triangle$ *abc* $\sim$ ABC (22). In the same manner construct $\triangle$ *acd* $\sim$ ACD, and *ade* $\sim$ ADE.

Then *abcde* $\sim$ ABCDE (28).

XLVIII.

Problem. *To construct a triangle equal to a given polygon.*

Let ABCDEF be the given polygon.



SOLUTION. Take three consecutive vertices, as B, C, D. Draw BD and CG || BD, meeting ED produced in G. Join BG.

Then $\triangle$ BGD = BCD, since they have the same base, BD, and are between the same parallels, BD and CG (2, Cor. 3).

Hence the polygon ABGEF = ABCDEF, and it has one less side.

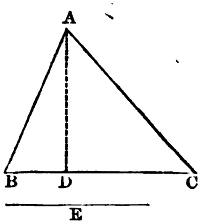
By taking the vertices B, G, E, a polygon may be constructed equal to ABGEF, and having one side less. By repeating this process, we shall finally obtain a triangle equal to the given polygon.

Complete the construction.

XLIX.

Problem. *To construct a square equal to a given triangle.*

Let ABC be a given triangle, BC its base, and AD its altitude. It is required to construct a square equal to ABC.



Solution. Find a mean proportional, E, between AD and $\frac{1}{2}$BC (43). The square described upon E will be equal to ABC; for

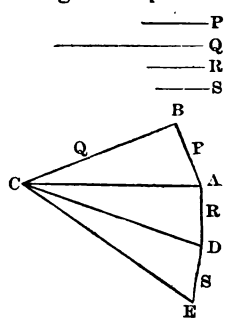
$$E^2 = AD \times \tfrac{1}{2}BC = ABC \text{ (7, Cor.)}.$$

Scholium. By this and the preceding problem a square may be constructed equal to any given polygon.

L.

Problem. *To construct a square equal to the sum of two or more given squares, or to the difference of two given squares.*

1. Let P, Q, R, S, &c., be the sides of given squares. It is required to find a line, EC, such that $EC^2 = P^2 + Q^2 + R^2 + S^2$.



Solution. Take AB = P, and from one extremity, B, draw BC $\perp$ AB, and = Q. Join AC; then

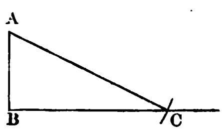
$$AC^2 = P^2 + Q^2 \text{ (12)}.$$

Draw AD $\perp$ AC, and = R. Join DC; then $DC^2 = AC^2 + R^2 = P^2 + Q^2 + R^2$.

By repeating this process we may find a square equal to the sum of any number of given squares.

2. It is required to construct a line, BC, such that

$$BC^2 = Q^2 - P^2.$$



Solution. Construct a right angle, ABC, and lay off AB = P. With A as a centre, and a radius equal to Q, draw an arc cutting BC in C. Join AC; then

$$BC^2 = AC^2 - AB^2 = Q^2 - P^2 \text{ (12, Cor. 1)}.$$

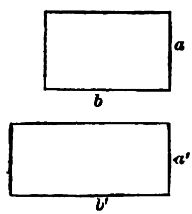
SCHOLIUM 1. By this and the preceding problems, a square may be constructed equal to the sum of any number of polygons, or to the difference of any two polygons.

SCHOLIUM 2. If P, Q, R, and S are homologous sides of similar polygons, the line EC (Fig. 1) is the homologous side of a similar polygon equal to the sum of those polygons (32). The line BC (Fig. 2) is the homologous side of a similar polygon equal to the difference of the polygons whose homologous sides are P and Q.

LI.

Problem. *Upon a given straight line to construct a rectangle equal to a given rectangle.*

Let b' be the given line, a the altitude, and b the base, of the given rectangle.



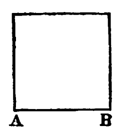
SOLUTION. Find a', the fourth proportional to b', a, and b (42). The rectangle constructed on b' with the altitude a' will be the rectangle required.

For, since $b' : b = a : a'$, we have,

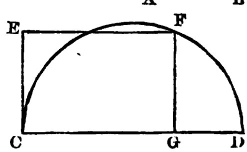
$$a' \times b' = a \times b \text{ (7, Cor.)}.$$

LII.

Problem. *To construct a rectangle when its area and the sum of two adjacent sides are given.*



Let AB be the side of the square equal to the given area, and let CD be equal to the sum of the two sides of the required rectangle. To find a point, G, in CD, such that $CG \times GD = AB^2$.



SOLUTION. Upon CD as a diameter describe a semicircle. From C draw $CE = AB$, and $\perp$ CD. From E draw EF $\parallel$ CD, cutting the circumference in F. From F draw FG $\perp$ CD. Then G is the point required.

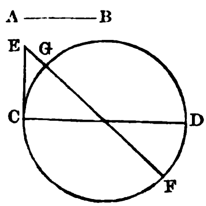
For $CG : FG = FG : GD$ (43) : hence $FG^2 = CG \times GD$;
but $FG = CE = AB$:
hence $AB^2 = CG \times GD$.

When is the solution impossible?

LIII.

Problem. *To construct a rectangle when its area and the difference of two adjacent sides are given.*

Let AB be the side of the square equal to the given area, and let CD be equal to the difference of the two sides of the required rectangle. To find two lines, EF and EG, such that $EF \times EG = AB^2$, and $EF - EG = CD$.



Solution. Upon CD as a diameter describe a circle. At C draw the tangent $CE = AB$, and from E the secant EF through the centre of the circle. Then EF and EG are the base and altitude of the required rectangle; for

$$EF \times EG = EC^2 = AB^2 \text{ (37, Cor. 2)};$$

and $$EF - EG = GF = CD.$$

Construct the rectangle.

Exercise 1. If, from a point within a parallelogram, lines be drawn to the four vertices, two opposite triangles thus formed are together equal to one-half the parallelogram.

Exercise 2. The triangle formed by joining the middle point of one of the non-parallel sides of a trapezoid to the extremities of the opposite side is equal to one-half the trapezoid.

Exercise 3. Given any triangle, to construct an isosceles triangle of equal area, whose vertical angle is an angle of the given triangle (6).

Exercise 4. Bisect a given triangle by a line parallel to one of its sides.

BOOK IV.

REGULAR POLYGONS AND THE CIRCLE.

I.

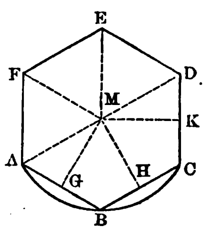
DEFINITION. A **regular polygon** is a polygon which is both equilateral and equiangular.

II.

Theorem. *A circle may be circumscribed about, or inscribed within, any regular polygon.*

HYPOTH. ABCDEF is a regular polygon.

TO BE PROVED. 1. A circumference may be made to pass through the vertices A, B, C, D, E, and F; that is, circumscribed about the polygon.



PROOF. From G and H, the middle points of AB and BC, erect perpendiculars: they will meet in a point M (I., 13), which is equally distant from A, B, and C (I., 31). From M as a centre describe the arc ABC; draw the lines MA and MD; about the line MH revolve the quadrilateral MABH until BH falls upon its equal line, HC. The line BA will take the direction CD, for the angle $B = C$; the point A will fall on D, since $BA = CD$; and the line MA will coincide

with and be equal to MD: hence D is a point of the circumference which passes through A, B, and C.

In like manner it may be shown that the same circumference will pass through the remaining vertices, E and F.

To be Proved. 2. A circle may be inscribed within the polygon ABCDEF.

Proof. The sides AB, BC, CD, &c., are equal chords of the circumscribed circle. Hence $MG = MH = MK$, &c. (II., 5), and a circumference drawn from M as a centre, with a radius MG, will touch the sides, and be inscribed in the polygon (II., 11, Cor. 7).

III.

Def. 1. The centre, M, of the circle circumscribed about a regular polygon is called the **centre of the polygon.**

Def. 2. The line MA is the **radius of the polygon.**

Def. 3. The perpendicular MG is the **apothegm.**

Def. 4. The angle AMB is called the **angle at the centre** of the polygon.

Cor. 1. The angle at the centre of a regular polygon is equal to $\frac{4R}{n}$, where n is the number of sides of the polygon.

Cor. 2. The angle at the centre of a regular hexagon is $\frac{4R}{6} = \frac{1}{3}$ of 2R. Consequently, in the isosceles triangle MAB, each of the equal angles, MAB and MBA, is $\frac{1}{3}$ of 2R (I., 17, Cor. 5). Hence the triangle is equiangular and equilateral; that is, *the side of a regular hexagon is equal to its radius.*

Cor. 3. The angle of any regular polygon is the supplement of its angle at the centre; for by I., 35, the angle of the polygon $= \frac{2nR - 4R}{n} = 2R - \frac{4R}{n} = 2R -$ angle at the centre.

IV.

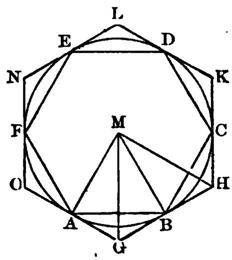
Theorem. *If a circumference be divided into any number of equal parts, the chords joining the successive points of division will form a regular inscribed polygon.*

For, 1st, the sides will be equal, being chords of equal arcs (II., 6); and, 2d, the angles will be equal, being inscribed in equal segments (II., 11, Cor. 1).

V.

Theorem. *If, through the vertices of a regular inscribed polygon, tangents to the circumference be drawn, they will form a regular circumscribed polygon.*

HYPOTH. ABCDEF is a regular inscribed polygon, and OG, GH, HK, . . . are tangents at the points A, B, C, . . .



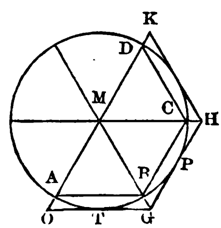
TO BE PROVED. GHKLNO is a regular polygon (1).

PROOF. In the triangles GAB, HBC, . . . ∠GAB = GBA = HBC = HCB, = . . . since each has the same measure as one-half the equal arcs AB, BC, . . . (II., 11, Cor. 5).

Also AB = BC, = . . . : hence GAB, HBC, are equal isosceles triangles (I., 21), and AG = GB = BH = HC = CK, &c., which gives GH = HK = . . .

Also ∠AGB = BHC = CKD = . . . and the circumscribed polygon GHKLNO is regular.

COR. 1. Since △AMG = GMB = BMH = . . . it follows that ∠AMG = GMB = BMH = . . . ; that is, the lines MA, MG, MB, MH, . . . divide the circumference into equal arcs (II., 6). If the circumscribed polygon be turned about the circle through one of these arcs, the radius MG will pass through the point B, MH through C, &c.; also GH will be parallel to BC, HK to CD, &c., since BC will cut the sides MG and MH proportionally (III., 17). The tangent point of GH will be the centre of the arc BC.

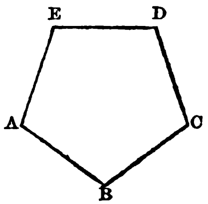


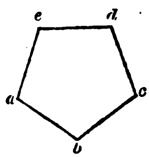
COR. 2. If the chords AT, TB, BP, &c., be drawn, a regular inscribed polygon of double the number of sides will be formed.

If through the points A, B, C, &c., tangents be drawn, intersecting the tangents OG, GH, &c., a regular circumscribed polygon of double the number of sides will be formed.

VI.

Theorem. *Regular polygons of the same number of sides are similar.*





HYPOTH. ABCDE and *abcde* are two regular polygons of the same number of sides.

TO BE PROVED. ABCDE $\sim$ *abcde*.

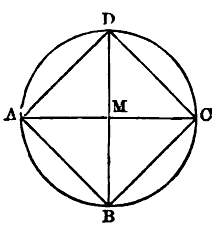
PROOF. 1. The polygons are mutually equiangular, since the size of the angles depends on the number of sides (I., 35).

2. The corresponding sides are proportional, since the sides in each are equal: hence the polygons are similar (III., 20).

VII.

Problem. *To inscribe a square in a given circle.*

Draw two diameters, AC and BD, at right angles to each other. They will divide the circumference into four equal parts (II., 6, Cor. 2). Draw the chords AB, BC, CD, and DA. Then ABCD is the inscribed square required (4).



SCHOLIUM. In the right-angled triangle AMB, $AB^2 = AM^2 + BM^2 = 2AM^2$,

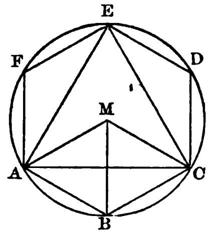
and $AB = AM\sqrt{2} = r\sqrt{2}$;

or, $AB : r = \sqrt{2} : 1$.

VIII.

Problem. *To inscribe a regular hexagon in a given circle.*

Apply the radius, AM, to the circumference six times as a chord. It will form the hexagon ABCDEF required (3, Cor. 2).



Cor. By joining the alternate vertices, A, C, and E, an equilateral triangle will be formed.

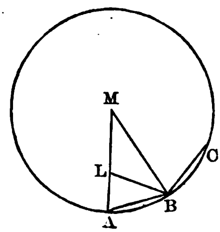
Scholium. Draw the radii, MA, MB, and MC: then AMCB is a rhombus, and $4AM^2 = MB^2 + AC^2$ (III., 16). Subtracting $AM^2 = MB^2$, we have,

$$3AM^2 = AC^2;\ \text{or, } AC = AM\sqrt{3} = r\sqrt{3}.$$

IX.

Problem. *To inscribe a regular decagon in a given circle.*

Divide the radius, MA, in extreme and mean ratio (III., 44). Apply ML, the greater part, to the circumference ten times as a chord: it will form a regular inscribed decagon.



Proof. Draw MB and BL. Then, since $AB = ML$, we have, $AM : AB = AB : AL$ (III., 44); that is, in the two triangles, AMB and ABL, the sides about the common angle A are proportional: hence $AMB \sim ABL$ (III., 23). But $\triangle$ AMB is isosceles: hence $AB = BL = LM$; and $\triangle$ LMB is also isosceles. Then $\angle ALB = MAB = LMB + LBM = 2M$ (I., 18), and $\angle MAB + MBA = 4M$.

Adding $\angle M$ to both members, and remembering that

$$\angle M + MAB + MBA = 2R \text{ (I., 17)},$$

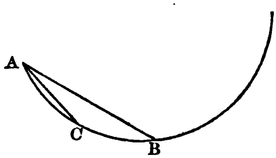
we have $$2R = 5M;\ \text{or, } \angle M = \frac{2R}{5} = \frac{4R}{10}.$$

Hence each of the arcs AB, BC, &c., is $\frac{1}{10}$ of the circum-

ference, and $ML = AB$ is the side of a regular inscribed decagon (4).

Scholium. $AB = \frac{AM}{2}(\sqrt{5} - 1)$ (III., 44, Schol.).

Cor. 1. By joining the alternate vertices, a regular inscribed pentagon will be formed.



Cor. 2. If AB is the side of a regular hexagon, and AC the side of a regular decagon, then CB will be the side of a regular pentadecagon; for $\frac{1}{6} - \frac{1}{10} = \frac{1}{15}$; and these fractions express the values of these arcs respectively.

X.

1. If we bisect the arcs of an inscribed square, we may form regular inscribed and circumscribed polygons of 8, 16, 32, &c., sides. In like manner we may obtain from the hexagon regular polygons of 12, 24, 48, &c., sides; from the decagon, regular polygons of 20, 40, 80 sides; and from the pentadecagon, regular polygons of 30, 60, 120, &c., sides.

In addition to these, Gauss, in his "Disquisitiones Arithmeticæ," showed that regular polygons of $2^n + 1$ sides may be constructed by means of the circle and straight lines, when n is an integer, and $2^n + 1$ a prime number; as, 17, 257.

2. It is evident that any circumscribed polygon is larger than one of double the number of sides, and that, as the number of sides is increased, the area of the polygon approaches that of the circle, but will never pass that limit. So long as the sides have a finite length, the polygon will be larger than the circle; but when the sides are infinitely small, that is, after the number of sides has been doubled an infinite number of times, it will correspond with the circle.

3. In like manner, it may be shown that any inscribed polygon is less than that of double the number of sides, and

that its limit is likewise the circle. Hence *the circle may be considered a regular polygon of an infinite number of sides.* It is evident that the limit of the apothegm is the radius.

XI.

Let S_n represent the side of a regular inscribed polygon of n sides, r the radius, a the apothegm, P the perimeter, and A_n the area. Then

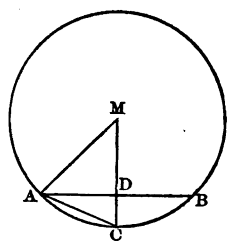
$$S_4 = r\sqrt{2} \ (7),$$
$$S_6 = r \ (3, \text{Cor. } 2),$$
$$S_{10} = \frac{r}{2}(\sqrt{5}-1) \ (9, \text{Schol.}),$$
$$S_3 = r\sqrt{3} \ (8).$$

Also $$P = nS_n, \ A_n = \frac{nS_na}{2} = \frac{Pa}{2};$$

for if radii be drawn to all the vertices, the polygon will be divided into n equal triangles, and the area of each will be $\frac{S_na}{2}$ (III., 7, Cor.).

XII.

Problem. *Given* S_n *to find* S_{2n}.



Let $AB = S_n$, $AC = S_{2n}$, $MD = a$, $DC = r - a$. Then, in the triangle ADC,

$AC^2 = AD^2 + DC^2$ (III., 12) ; or,

$$S_{2n}^2 = \frac{S_n^2}{4} + r^2 - 2ar + a^2.$$

But $\triangle$ ADM gives $a^2 = r^2 - \frac{S_n^2}{4}$. Substituting this value of a, and reducing,

we have, $$S_{2n} = \sqrt{2r^2 - r\sqrt{4r^2 - S_n^2}},$$

and $$S_{2n} = r\sqrt{2 - \sqrt{4 - \frac{S_n^2}{r^2}}}.$$

COR. 1. Applying this formula to the inscribed square, and taking the perimeter instead of the side, we have,

$$S_4 = r\sqrt{2}\ (11).$$

$$S_8 = r\sqrt{2-\sqrt{4-\frac{S_4^2}{r^2}}},$$

$$S_{16} = r\sqrt{2-\sqrt{4-\frac{S_8^2}{r^2}}},$$

. . . .

$$nS_n = P_n = nr\sqrt{2-\sqrt{4-\frac{S_{\frac{n}{2}}^2}{r^2}}}. \qquad \text{I.}$$

Also r is the apothegm of the circumscribed polygon; and the side of the inscribed polygon, S_n, is to the side of the similar circumscribed pologon, S_n^c, as a is to r (III., 26): hence

$$P_n : P_n^c = a : r, \text{ and}$$

$$P_n^c = \frac{rP_n}{a}; \text{ but } a = \sqrt{r^2 - \frac{S_n^2}{4}} : \text{ hence}$$

$$P_n^c = \frac{rP_n}{\sqrt{r^2 - \frac{S_n^2}{4}}} = \frac{2rP_n}{\sqrt{4r^2 - S_n^2}}. \qquad \text{II.}$$

Let $n = 4096$, and $2r = d = 1$; then I. and II. give

$$P_{4096} = 3.141592, \text{ and}$$

$$P_{4096}^c = 3.141593;$$

but the circumference, C, is greater than the perimeter of the inscribed polygon, and less than that of the circumscribed polygon: hence $C = 3.141592$ is correct to within *one-millionth* of the diameter d, which was taken as the unit.

COR. 2. From I. it is evident that if the radius, or diameter, were increased, the value of P_{4096} would be increased in the same ratio, and consequently the circumference C: hence, for any circle whose diameter is d, we have,

$$C = 3.14159 \times d = \pi d,$$

where π denotes not the approximate value 3.14159, but the *exact ratio* of the diameter to the circumference.

COR. 3. It was shown (11) that the area of a regular polygon $= \frac{Pa}{2}$; but the circle is a regular polygon whose

perimeter is denoted by C, and apothegm by r (10): hence

area of a circle $= \dfrac{Cr}{2} = \dfrac{\pi d r}{2} = \pi r^2.$

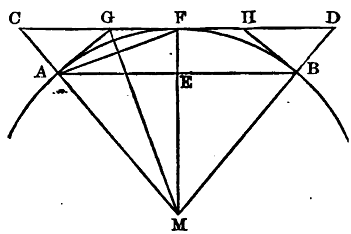
COR. 4. The area of a sector $= \dfrac{\text{arc} \times \text{radius}}{2}$.

SCHOLIUM. It may be shown by higher analysis that π is irrational; and hence the circumference and area of a circle cannot be expressed by numbers in terms of the radius. There is no known method of constructing a square *exactly* equal to a circle, or a line equal to a circumference; nor has the impossibility of this problem been demonstrated. See (14.)

XIII.

Problem. *The area of a regular inscribed polygon being given, and that of a similar circumscribed polygon, to find the areas of the regular inscribed and circumscribed polygons of double the number of sides.*

Let p denote the area of the regular inscribed polygon; P, that of the similar circumscribed polygon; p_1 and P_1, the areas of the regular inscribed and circumscribed polygons of double the number of sides; and n, the number of sides of the given polygon. Let the chord AB be the side of the given inscribed, and the tangent CD that of the given circumscribed polygon. Draw the chord AF, and through A and B the tangents AG and BH; then AF and GH are the sides of the required polygons (5, Cor. 2). Also the area



$$\text{AEM} = \frac{p}{2n},\quad \text{CFM} = \frac{P}{2n},\quad \text{AFM} = \frac{p_1}{2n}, \text{AGFM} = \frac{P_1}{2n},$$

$$\text{ACG} = \frac{P - P_1}{2n}, \text{AGF} = \frac{P_1 - p_1}{2n}.$$

1. Then, AEM : AFM = ME : MF (III., 4) = MA : MC (III., 17) = AFM : CFM: hence $p : p_1 = p_1 : P$,

and $p_1 = \sqrt{pP}$. I.

2. ACG : AGF = CG : GF (III., 4) = CMG : GMF = CMG : GMA = CM : AM = CFM : AFM.

Substituting the above values in the first and last couplets, and omitting $2n$, we have,

$$P - P_1 : P_1 - p_1 = P : p_1 ; \text{ or, } \frac{1}{P_1} = \frac{1}{2}\left(\frac{1}{p_1} + \frac{1}{P}\right) \text{ II.};$$

that is, p_1 is a *geometrical mean* between p and P, and P_1 is a *harmonical mean* between p_1 and P.

Scholium. Applying formulas I. and II. to squares inscribed in and circumscribed about a circle whose radius is unity, we have for the areas of the inscribed and circumscribed octagons, $p_1 = \sqrt{8} = 2.8284271$,

$$\frac{1}{P_1} = \frac{1}{2}\left(\frac{1}{\sqrt{8}} + \frac{1}{4}\right) = .30177669,\ P_1 = 3.3137085.$$

Doubling the number of sides a second time, we obtain for the regular polygons of 16 sides,

$$p_2 = \sqrt{p_1 P_1} = 3.0614674,$$

$$\frac{1}{P_2} = \frac{1}{2}\left(\frac{1}{p_2} + \frac{1}{P_1}\right) = .314208+,\ P_2 = 3.1825979.$$

By continuing an application of these formulas, we may obtain the results given in the following

TABLE.

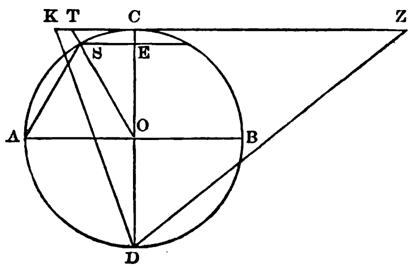
No. of Sides.	Area of Inscribed Polygon.	Area of Circumscribed Polygon.
4	2.0000000	4.0000000
8	2.8284271	3.3137085
16	3.0614674	3.1825979
32	3.1214451	3.1517249
64	3.1365485	3.1441184
128	3.1403311	3.1422236
256	3.1412772	3.1417504
512	3.1415138	3.1416321
1024	3.1415729	3.1416025
2048	3.1415877	3.1415951
4096	3.1415914	3.1415933
8192	3.1415923	3.1415928

From the last result it is evident that the inscribed and circumscribed polygons of 8192 sides differ from each other by less than one-millionth of the square on the radius which was taken as the unit: hence the approximate area of a circle with the radius 1 is 3.141592 (12, Cor. 3.) The value $\pi = 3.1416$ is sufficiently accurate for ordinary calculations.

XIV.*

Problem.† *To construct a straight line which shall approximately equal a semi-circumference; also to construct a triangle which shall approximately equal the area of a circle.*

SOLUTION. 1. Through C, a point of the circumference, draw a tangent, TZ; and the diameters CD and AB || TZ. Draw the chord AS = radius, and the line OST. Make TZ = 3 times the radius, and join DZ. Then DZ is about $\frac{1}{17000}$ of the radius shorter than the semi-circumference ACB.



PROOF. AS and SF are the sides of a regular inscribed hexagon (3, Cor. 2): hence TC = one-half the side of a regular circumscribed hexagon (5, Cor. 1). The right-angled triangle OSE gives $OE = \sqrt{OS^2 - SE^2} = \sqrt{1 - \frac{1}{4}} = \frac{1}{2}\sqrt{3}$; but OE : OC = SE : TC; or, $\frac{1}{2}\sqrt{3} : 1 = \frac{1}{2} : TC$: hence

$TC = \frac{1}{\sqrt{3}}$ when $r = 1$. $CZ = TZ - TC = 3 - \frac{1}{\sqrt{3}}$, and $CD = 2$:

hence $DZ = \sqrt{CD^2 + CZ^2} = \sqrt{4 + \left(3 - \frac{1}{\sqrt{3}}\right)^2} = \sqrt{13\frac{1}{3} - 2\sqrt{3}}$ $= 3.141533$ (III., 12). But the semi-circumference ACB $= 3.141592$ when r is the unit (12, Cor. 1): hence DZ =

† Crelle's Journal für die reine und angewandte Mathematik Bd. 32, S. 91.

semi-circumference ACB to within .00006; or, $\frac{1}{17000}$ of the radius.

Solution. 2. Take $ZK = DZ$, and draw DK; then the triangle $ZDK =$ area of the circle ACBD to within $\frac{1}{17000}$ of that area.

Proof. $\triangle ZDK = \frac{1}{2}ZK.CD = ZK$ (III., 7, Cor.);
also circle $ACBD = 2ACB.\frac{1}{2}AO = ACB$ (12, Cor. 3).

But $ZK = DZ = ACB$ to within $\frac{1}{17000}$.

Hence $\triangle ZDK =$ area of circle ACBD to within $\frac{1}{17000}$ of that area.

SOLID GEOMETRY.

BOOK V.

STRAIGHT LINES AND PLANES.

I.

THROUGH two points an infinite number of planes may be made to pass; for a plane in which the two points lie will take an infinite number of positions, if turned about the line joining those points.

A third point not in this connecting line fixes one position of the plane: hence the position of a plane is fixed, —

By three points not in the same straight line,

By a straight line and a point without it,

By two straight lines which cut each other, or,

By two parallel lines; for it follows from their definition that two parallel lines lie in the same plane (I., 8).

COR. 1. Two planes which have three points common that are not in the same straight line fall together, and form one plane.

COR. 2. Hence, if two planes cut each other, all their common points lie in the same straight line; that is, their common section is a straight line.

DEF. 1. Two planes are **parallel** when they have no points common, however far produced.

If a straight line has two points common with a plane, it lies wholly in that plane (Gen. Prin. 12): hence a straight

line [plane] lies wholly in a plane; has one point [straight line] common with a plane, and is said to cut it; or has no point common with a plane, however far produced, and is said to be parallel to it.

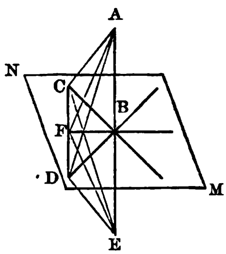
Def. 2. When a straight line so cuts a plane that it is perpendicular to all the lines of the plane that pass through the common point, it is said to be **perpendicular** to the plane, and the plane is perpendicular to the line.

LINES WHICH CUT A PLANE.

II.

Theorem. *If a straight line is perpendicular to two lines of a plane at their point of intersection, it is perpendicular to the plane.*

Hypoth. AB is perpendicular to BC and BD, which lie in the plane MN.



To be Proved. AB is perpendicular to any other line, BF, of the plane, which passes through B, and, consequently, to the plane itself.

Proof. Prolong AB until BE = AB. Draw any line, CD, cutting BF in F. From A and E draw lines to C, F, and D.

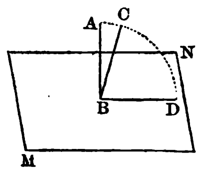
Then $\triangle ABC \cong EBC$: hence $AC = EC$ (I., 20).
$\triangle ABD \cong EBD$: hence $AD = ED$ (I., 20).
Therefore $\triangle ACD \cong ECD$: hence $\angle ACF = ECF$ (I., 24),
and $\triangle ACF \cong ECF$: hence $AF = EF$ (I., 20);
then $\triangle ABF \cong EBF$ (I., 24):
hence $\angle ABF = EBF = R$ (I., 4, Cor. 1): that is,
$AB \perp MN$.

The following demonstration was given by Legendre:

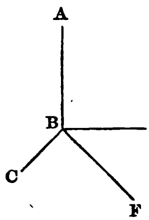
Let the line CD be so drawn that $CF = FD$ (III., 46);
then $AC^2 + AD^2 = 2AF^2 + 2CF^2$ (III., 15),
and $BC^2 + BD^2 = 2BF^2 + 2CF^2$,

which give $AC^2 - BC^2 + AD^2 - BD^2 = 2AF^2 - 2BF^2$; or,
$AB^2 + AB^2 = 2AF^2 - 2BF^2$ (III., 12, Cor. 1), which gives
$AB^2 = AF^2 - BF^2$: hence $\angle ABF = R$ (III., 14, Cor.);
that is, $AB \perp MN$.

COR. 1. From a point, A, but one perpendicular can be drawn to the plane; for if it were possible to draw two, AB and AF, the triangle ABF would contain two right angles, which is impossible.



COR. 2. From a point, B, in the plane, but one perpendicular can be erected. For, if possible, let BA and BC be perpendicular to MN, and let BD be the intersection of the plane ABC (1) with MN: then ABD and CBD are each right angles, and equal to each other, which is impossible.

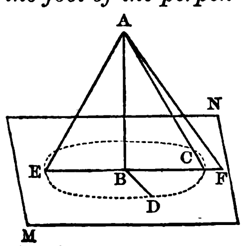


COR. 3. Two or more perpendiculars, BC, BF, BD, &c., to the same point, B, of the line AB, lie in one plane perpendicular to AB (proof indirect); and if a right-angled triangle, ABD, be turned about AB, the line BD will generate a plane $MN \perp AB$.

COR. 4. The perpendicular AB is shorter than any oblique line, AD, and measures the distance of the point A from the plane MN.

III.

Theorem. *All oblique lines drawn from the same point, and meeting a plane at equal distances from the foot of the perpendicular, are equal; and, of two oblique lines meeting the plane at unequal distances from the foot of the perpendicular, the more remote is the greater.*



HYPOTH. 1. $AB \perp MN$, and $BC = BD = BE, =$, &c.

TO BE PROVED. $AC = AD = AE, =$, &c.

PROOF. $\triangle ABC \cong ABD \cong ABE =$, &c. (I., 20):
hence $AC = AD = AE =$, &c.

HYPOTH. 2. $BF > BC$ or BD.

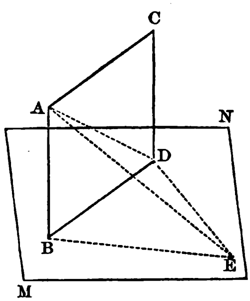
TO BE PROVED. $AF > AC$ or AD. (See I., 30, 4.)

COR. The locus of the extremities, C, D, E, &c., of the equal oblique lines, is a circumference whose centre is the foot of the perpendicular AB: hence to draw a perpendicular to a plane, MN, from a point, A, find three points, C, D, E, of the plane, equally distant from A. The centre of the circumference drawn through these points will be the foot of the perpendicular.

IV.

Theorem. *If one of two parallel lines is perpendicular to a plane, the other is perpendicular to the same plane.*

HYPOTH. $AB \parallel CD$, and $AB \perp MN$.

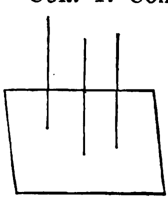


TO BE PROVED. $CD \perp MN$.

PROOF. Through AB and CD pass a plane cutting MN in the line BD. Draw $DE \perp BD$ in the plane MN, and make $DE = AB$. Draw AD, AE, and BE.

Then $\triangle ABD \cong EDB$; hence $AD = BE$ (I., 20): and $\triangle ABE \cong ADE$; hence $\angle ABE = ADE = R$ (I., 24). Therefore DE, being perpendicular to BD and AD, is perpendicular to their plane ABDC: hence $DE \perp DC$ (2). But $CD \perp BD$ (I., 10, Cor.): hence CD is perpendicular to the plane of BD and DE; or, $CD \perp MN$.

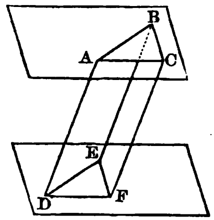
COR. 1. Conversely, *two straight lines*, AB *and* CD, *perpendicular to the same plane*, MN, *are parallel to each other.* Proof indirect (2, Cor. 2).



COR. 2. *Two straight lines that are parallel to a third straight line are parallel to each other;* for they are each perpendicular to a plane that is drawn perpendicular to the third line.

V.

Theorem. *If two angles not in the same plane have their sides parallel each to each, and lying in the same direction, they are equal.*



Hypoth. AB || DE, and AC || DF.

To be Proved. ∠ BAC = EDF.

Proof. Take AB = DE, and AC = DF. Draw BC, EF, AD, BE, and CF. Then AB = || DE : hence AD = || BE (I., 39); also AC = || DF : hence AD = || CF (I., 39), and BE = || CF (4., Cor. 2) : hence BC = || EF.

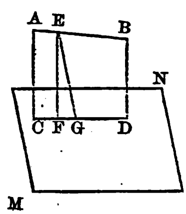
Hence △ BAC ≅ EDF (I., 24), and ∠ BAC = EDF.

Scholium. Two parallel sides, AB and DE, lie in the same direction when they are on the same side of the line AD joining their vertices. (See I., 12, and Cor.)

VI.

Theorem. *The locus of all the perpendiculars drawn from a line to a plane is a plane.*

Hypoth. AC, EF, and BD are three lines drawn from AB perpendicular to the plane MN.



To be Proved. EF lies in the plane of AC and BD.

Proof. If EF is not in the plane ACDB, draw EG in that plane parallel to AC; then EG ⊥ MN (4); but EF ⊥ MN (Hypoth.). That is, from the point E two perpendiculars are drawn to the plane, which is impossible (2, Cor. 1) : hence EF lies in the plane ABDC, and a perpendicular drawn from any other point of AB will lie in the same plane. That is, the locus of all the perpendiculars drawn from AB to MN is the plane ABDC.

Def. 1. The **projection of a point**, E, upon a plane,

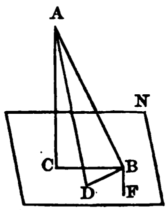
MN, is F, the foot of the perpendicular let fall from the point upon the plane. The **projection of a line,** AB, upon a plane, MN, is CD, the projection of all its points upon that plane.

DEF. 2. The acute angle which a straight line makes with its projection upon a plane is called the **inclination** of the line to the plane; and the plane of these lines, the **plane of inclination,** or **projecting plane.**

COR. It is evident that the projection of a straight line upon a plane is a straight line.

VII.

Theorem. *The inclination of a line to a plane is the smallest angle which the line makes with any line of the plane.*



HYPOTH. AC $\perp$ MN, and BC is the projection of AB on MN. BD is any other line of the plane.

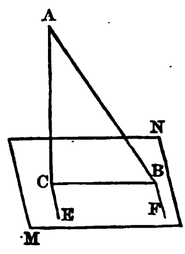
TO BE PROVED. $\angle$ ABC $<$ ABD.

PROOF. Take BD $=$ BC; draw AD.

Then AC $<$ AD (2, Cor. 4): hence

$\angle$ ABC $<$ ABD (I., 23).

COR. The angle adjacent to the inclination of a line to a plane is the greatest angle which that line makes with any line of the plane. If BD be turned about the point B to the position BF $\perp$ BC, then $\angle$ ABF will be an arithmetical mean between the inclination and its adjacent angle; that is, $\angle$ ABF $=$ R. For draw CE $\parallel$ BF; then CE $\perp$ BC and AC, that is, to the plane ABC (2): hence BF $\perp$ plane ABC (4), and BF $\perp$ AB.



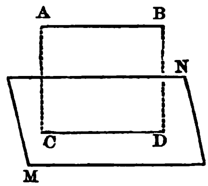
LINES PARALLEL TO A PLANE.

VIII.

Theorem. *A line is parallel to a plane, when it is parallel to a line of that plane.*

HYPOTH. CD lies in the plane MN, and AB || CD.

TO BE PROVED. AB || MN.

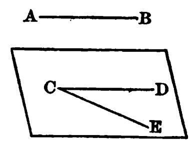


PROOF. Pass a plane through AB and CD. Then, if AB can meet the plane MN, it must meet it in their intersection, CD; but AB || CD: hence AB || MN.

COR. 1. If MN be turned about CD, the demonstration will be the same for every position of the plane: hence *one of two parallel lines is parallel to every plane drawn through the other.*

COR. 2. If two planes, AD and MN, cut each other, a line, AB, in one, parallel to their intersection, CD, is parallel to the other plane; and, conversely, if AB in one is parallel to the other plane, it is parallel to their intersection, CD.

COR. 3. A line parallel to the intersection of two planes is parallel to those planes.



COR. 4. Through one of two straight lines not parallel to each other, one plane may be passed parallel to the other straight line. For, if AB and CE are the lines, draw CD || AB: then the plane DCE || AB.

COR. 5. Through any point, P, a plane may be passed parallel to the plane DCE. It will be parallel to the line AB. The plane P is the locus of all the lines drawn through the point P parallel to the plane DCE.

From the above corollaries we have,

1. Through any point in space, a line may be drawn parallel to two given planes; and a plane may be drawn parallel to two given lines.

2. Through any point in space, an infinite number of lines

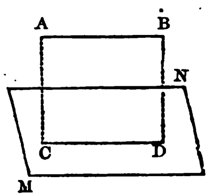
may be drawn parallel to a given plane; and an infinite number of planes may be drawn parallel to a given line.

IX.

Theorem. *A line parallel to a plane is everywhere equally distant from that plane.*

Hypoth. AB || MN.

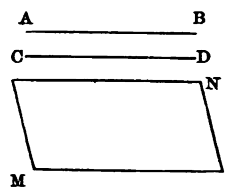
To be Proved. Any two lines, AC and BD, drawn from AB perpendicular to MN, are equal.



Proof. The perpendiculars AC and BD lie in the same plane (6). Also ABDC is a parallelogram, since AC || BD (4, Cor. 1), and AB || CD (8, Cor. 2): hence AC = BD (I., 37, Cor. 1); and AB is everywhere equally distant from MN.

X.

Theorem. *If one of two parallel lines is parallel to a plane, the other is parallel to the same plane.*



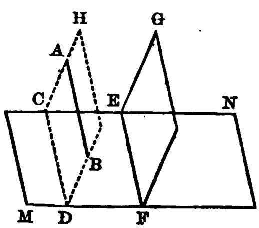
Hypoth. AB || CD, and AB || MN.

To be Proved. CD || MN.

Proof. Through AB and CD pass a plane. If this plane cuts MN in a line, EF, then AB || EF (8, Cor. 2); hence CD || EF || MN (4, Cor. 2): if it does not cut MN; then CD will not cut MN; that is, CD || MN.

XI.

Theorem. *If a straight line is parallel to each of two planes that cut each other, it is parallel to their intersection.*



Hypoth. AB || MN, and AB || FG.

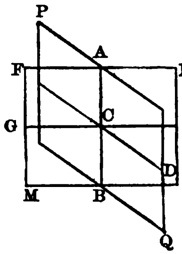
To be Proved. AB || EF, the intersection of MN and FG.

Proof. Through AB pass a plane, DH, cutting MN in CD; then AB || CD (8, Cor. 2), and CD || FG (10): hence CD || EF (8, Cor. 2), and AB || EF (4, Cor. 2.).

PLANES WHICH CUT EACH OTHER.— DIHEDRAL ANGLES.

XII.

DEF. 1. When two planes cut each other, they divide space into four parts, and form four angles, called **dihedral angles**, or **dihedrals**.



Thus MN and PQ cut each other in the line AB, forming four dihedrals, which are designated as follows, $Q\overline{AB}N$, $N\overline{AB}P$, $P\overline{AB}M$, and $M\overline{AB}Q$.

DEF. 2. The planes QAB and NAB are the **faces**, and AB the **edge**, of the angle $Q\overline{AB}N$.

DEF. 3. If from any point, C, of the edge AB, the perpendiculars CD and CE be erected in the faces, the angle DCE is called the **plane angle**, and the plane DCE, the **plane of divergence**, of the dihedral $Q\overline{AB}N$. The plane angle is the same for all points of AB (5).

COR. 1. The plane of divergence is perpendicular to the edge of a dihedral (2).

COR. 2. A plane perpendicular to the edge of a dihedral is the plane of divergence.

XIII.

Theorem. *Dihedrals have the same ratio as their plane angles.*

Demonstration like that of II., 9.

COR. 1. A dihedral has the same measure as its plane angle.

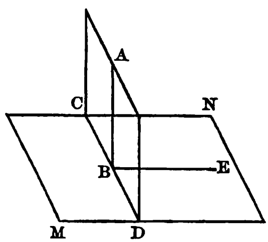
COR. 2. If DC and EC be prolonged, the angle GCF will be the plane angle of the vertical dihedral, $M\overline{AB}P$; and DCG, that of the adjacent dihedral, $M\overline{AB}Q$.

Hence vertical dihedrals are equal, and two adjacent dihedrals are equal to two right angles (I., 7).

Def. When the plane angle is a right angle, the two faces are said to be **perpendicular** to each other; and their angle is called a **right dihedral.**

XIV.

Theorem. *If a straight line is perpendicular to a plane, every plane passed through the line is perpendicular to that plane.*



Hypoth. AB $\perp$ MN.

To be Proved. Any plane, ACD, passed through AB, is perpendicular to MN.

Proof. Draw BE $\perp$ CD in the plane MN. Then $\angle$ABE $=$ R (Hypoth.) is the plane angle of the dihedral $\overline{\text{ACDN}}$: hence ACD $\perp$ MN (13, Cor. 1).

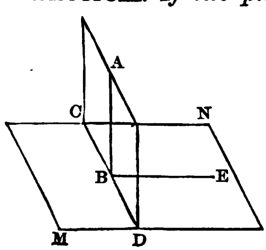
Cor. 1. If from the same point, B, three lines, BA, BE, and BC, are drawn perpendicular to each other, each line is perpendicular to the plane of the other two, and the three planes are perpendicular to each other.

Cor. 2. Two planes which cut each other are perpendicular to their plane of divergence; for their intersection is perpendicular to that plane.

Cor. 3. The plane of any line, and its projection upon a plane (6), is perpendicular to that plane.

XV.

Theorem. *If two planes are perpendicular to each other, a line in one of them perpendicular to their intersection is perpendicular to the other.*



Hypoth. ACD $\perp$ MN, and AB $\perp$ CD, their intersection.

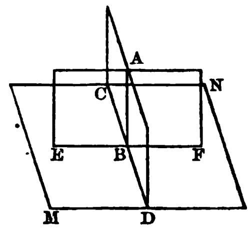
To be Proved. AB $\perp$ MN.

Proof. Draw BE $\perp$ CD in MN. Then $\angle$ ABE $=$ R (Hypoth.), since

it is the plane angle of ACD and MN; also ABD = R (Hypoth.): hence AB ⊥ MN (2).

Cor. 1. If ACD ⊥ MN, and from a point, B, of their intersection, BA be drawn perpendicular to MN, then BA lies in the plane ACD. Give proof in full (2, Cor. 2).

Cor. 2. If from a point, A, in ACD, one of two perpendicular planes, a line, AB, be drawn perpendicular to the other plane, MN, then AB lies in the plane ACD.

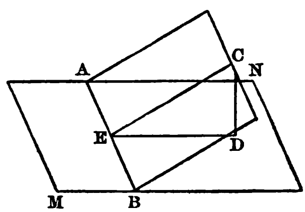


Cor. 3. If two planes, ACD and AEF, are perpendicular to MN, their intersection, AB, ⊥ MN. For from B erect BA ⊥ MN, it will lie in both the planes, and be their intersection (Cor. 1).

XVI.

Theorem. *If, from a point in one of two planes which cut each other, a perpendicular be drawn to the other, and from the foot of this a perpendicular be drawn to the intersection of the planes, these perpendiculars will lie in the plane of their divergence.*

Hypoth. CD ⊥ MN, and DE ⊥ AB, the intersection of CAB and MN.



To be Proved. CE ⊥ AB; that is, CED is the plane of divergence of MN and CAB.

Proof. CD ⊥ MN: hence the plane CDE ⊥ MN (14). Also EB ⊥ CED (15): hence CED is the plane of divergence (12, Cor. 2).

XVII.

Exercise 1. Given CE ⊥ AB, and CD ⊥ MN.
Prove that DE ⊥ AB (preceding figure).

Exercise 2. If a line, AB, is parallel to a plane, MN, any plane perpendicular to AB is perpendicular to MN.

Exercise 3. The points which are equally distant from three points of a plane are the locus of a line perpendicular to the plane.

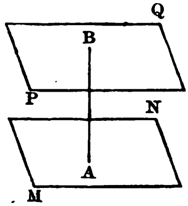
Exercise 4. If a line is perpendicular to a plane, every plane parallel to that line is also perpendicular to the plane.

PARALLEL PLANES.

XVIII.

Theorem. *If two planes are perpendicular to the same straight line, they are parallel to each other.*

Hypoth. MN and PQ are perpendicular to AB.

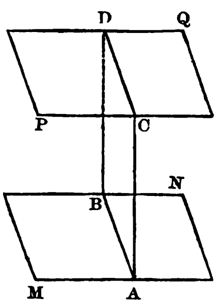


To be Proved. MN || PQ.

Proof. If MN and PQ are not parallel, they will meet, if produced. From O, a point in the line of their intersection, draw OA and OB; then ∠ OAB and OBA are right angles, which is impossible (I., 30): hence MN || PQ.

XIX.

Theorem. *The intersections of a plane with two parallel planes are parallel.*



Hypoth. MN || PQ. AB and CD are the intersections of ABDC with MN and PQ.

To be Proved. AB || CD.

Proof. If AB and CD are not parallel, they will meet, if produced, since they lie in the same plane; but, if they meet, the planes MN and PQ, in which they lie, will meet, which is impossible, because MN || PQ: hence AB || CD.

XX.

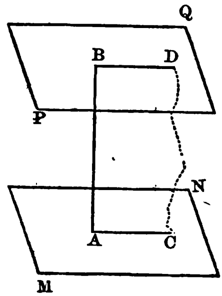
Theorem. *If one of two parallel planes is perpendicular to a straight line, the other is perpendicular to the same line.*

HYPOTH. MN $\parallel$ PQ, and MN $\perp$ AB.

TO BE PROVED. PQ $\perp$ AB.

PROOF. Through AB pass any plane intersecting MN and PQ in AC and BD. Then AC $\parallel$ BD (19); but AB $\perp$ AC (1, Def. 2): hence AB $\perp$ BD (I., 10, Cor.), and BD is any line in the plane PQ. Hence

PQ $\perp$ AB (1, Def. 2).



COR. 1. The line BD $\parallel$ MN; and the locus of all the lines drawn through B parallel to MN is a plane, PQ $\parallel$ MN. (See 8, Cor. 5.)

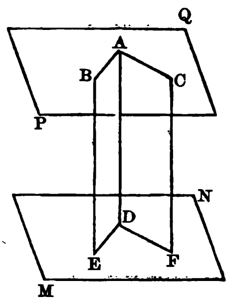
COR. 2. Two lines drawn through a point, B, parallel to a plane, MN, fix the position of a plane, PQ $\parallel$ MN.

XXI.

Theorem. *Two planes are parallel, when three points which fix the position of one are upon the same side of, and equally distant from, the other.*

HYPOTH. From the three points, A, B, C, the three lines, AD, BE, and CF, perpendicular to MN, are equal (2, Cor. 4).

TO BE PROVED. The plane PQ of the points A, B, and C, is parallel to MN.



PROOF. AD $\parallel$ BE $\parallel$ CF (4, Cor. 1). Through AD and BE pass the plane AE, and through AD and CF, the plane AF.

Then AD $=\parallel$ BE: hence AB $\parallel$ DE (1, 39). Also

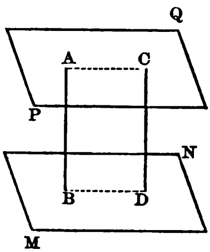
AD $=\parallel$ CF: hence AC $\parallel$ DF. Hence

AD $\perp$ AB and AC; that is, AD $\perp$ PQ (2), and PQ $\parallel$ MN (18).

COR. Two planes are parallel when the sides of an angle in one are respectively parallel to the sides of an angle in the other.

XXII.

Theorem. *Parallel lines included between two parallel planes are equal.*



Hypoth. MN || PQ, and AB || CD.

To be Proved. AB = CD.

Proof. Pass a plane through AB and CD, intersecting MN and PQ in BD and AC. Then AC || BD (19); but AB || CD (Hypoth.): hence AB = CD (I., 37, Cor. 1).

Cor. If the parallel lines are perpendicular to the planes, PQ and MN, they measure the distance of those planes from each other: hence *two parallel planes are everywhere equally distant.*

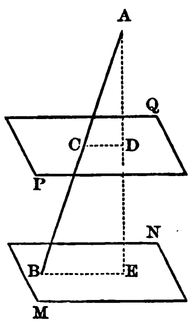
XXIII.

Theorem. *If two planes are parallel to a third plane, they are parallel to each other.*

For a line perpendicular to the third plane is perpendicular to each of the other two (20): hence the two planes are parallel to each other (18).

XXIV.

Theorem. *If a straight line cuts two parallel planes, the angles of inclination are equal.*



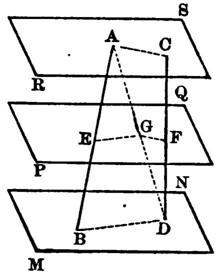
Hypoth. AB cuts the parallel planes, MN and PQ.

To be Proved. AB is equally inclined to MN and PQ.

Proof. Through a point, A, draw AE $\perp$ MN; then AD $\perp$ PQ (20). Through AB and AE pass a plane cutting MN and PQ in BE and CD; then BE || CD (19), and $\angle$ ACD = ABE (I., 10): that is, AB is equally inclined to MN and PQ (6, Def. 2).

XXV.

Theorem. *If two straight lines are cut by three parallel planes, they are divided proportionally.*



HYPOTH. The two lines AB and CD are cut by the three parallel planes, MN, PQ, and RS, in the points B, D, E, F, A, and C.

TO BE PROVED. AE : EB = CF : FD.

PROOF. Draw AD, and lay planes through BAD and ADC.

Then, since EG || BD (19),

$$AE : EB = AG : GD \text{ (III., 16) ; also,}$$

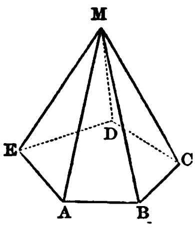
$$\text{since } GF \parallel AC,\ CF : FD = AG : GD :$$

$$\text{hence } AE : EB = CF : FD.$$

POLYHEDRAL ANGLES.

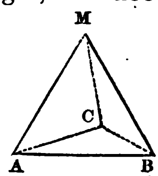
XXVI.

DEF. 1. Three or more planes which meet in a common point form a **polyhedral angle**; as, M—ABCDE. The term **polyhedral** will be used for polyhedral angle.



DEF. 2. The common point is called the **vertex** of the angle; the intersections of the planes, the **edges**; the portions of the planes between the edges, the **faces**; and the angles formed by the edges, the **face angles**; as, ∠AMB, BMC, &c.

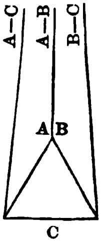
DEF. 3. A polyhedral which has but three faces is called a **trihedral**; as, M—ABC.



DEF. 4. A polyhedral is **convex** or **concave**, according as the polygon formed by the intersections of a plane, ABCD, with the faces, is convex or concave (I., 14, Def. 8).

XXVII.

Theorem. *If three planes cut each other in three lines, these lines are either parallel to each other, or will meet in one point if sufficiently produced.*



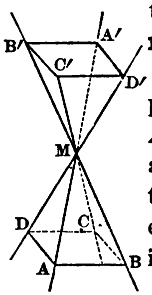
HYPOTH. A, B, and C are three planes whose intersections are designated by A—B, A—C, and B—C.

TO BE PROVED. A—B, A—C, and B—C are either parallel, or all meet in one point.

PROOF. *Case* 1. If A—B ∥ A—C, then A—B ∥ C (8): hence A—B ∥ B—C (8, Cor. 2), and the three lines are parallel to each other.

Case 2. If A—B is not parallel to A—C, then, since A—B lies in the plane B, and A—C in the plane C, their common point is in both those planes, that is, in B—C, their intersection; and hence the three lines meet in one point.

DEF. 1. If the edges of a polyhedral be produced through the vertex, the angle formed is called the **symmetrical polyhedral.**



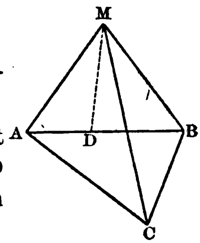
COR. 1. The face angles of the two symmetrical polyhedrals are equal each to each (I., 7); thus $\angle$ AMB = A′MB′, &c. But the polyhedrals are not congruent; for, if M—A′B′C′D′ be turned down so that MD′ fall upon MD, it is evident that the other edges are arranged in inverse order.

DEF. 2. If, from the vertex M, lines be drawn perpendicular to the faces, and planes be passed through them in the same order, the polyhedral formed is called the **polar polyhedral.**

COR. 2. The edges of a polyhedral are perpendicular to the faces of its polar angle; for any edge, BM, is perpendicular to the plane passed through the lines that are drawn perpendicular to the faces, AMB and BMC (2): hence every polyhedral is the polar angle of its polar polyhedral.

XXVIII.

Theorem. *The sum of any two face angles of a trihedral is greater than the third.*



Hypoth. M—ABC is a trihedral, of which $\angle$ AMB is the greatest face angle.

To be Proved. $\angle$ AMC + CMB > AMB.

Proof. In the face BMA construct $\angle$ BMD = BMC; draw AB, cutting MD in any point D; take MC = MD; and join AC and BC.

Then $\triangle$ BMD $\cong$ BMC (I., 20):

hence BD = CB.

But AC + CB > AB (I., 16):

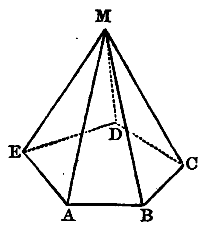
hence AC > AD, and $\angle$AMC > AMD (I., 23);

also $\angle$ CMB = BMD.

hence AMC + CMB > AMB.

XXIX.

Theorem. *The sum of the face angles of any convex polyhedral is less than four right angles.*



Hypoth. M—ABCDE is a convex polyhedral.

To be Proved. $\angle$ AMB + BMC + CMD +, &c., < 4R.

Proof. Pass a plane through the faces, forming a polygon, ABCDE, of n sides, when n is the number of faces.

Then $\angle$ ABC + BCD + CDE +, &c., = $2n$R—4R (I., 35),

and $\angle$ MBA + MBC > ABC,

$\angle$ MCB + MCD > BCD, &c., (28).

Adding these, and substituting, we have

MBA + MBC + MCB + MCD +, &c., > $2n$R — 4R.

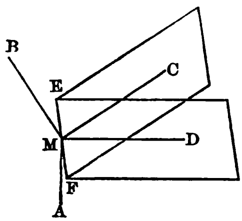
If, to the first member of this inequality, we add the face angles AMB, BMC, &c., we have all the angles of the

triangles AMB, BMC, &c., which $= 2n$R (I., 17): hence
$$\angle \text{AMB} + \text{BMC} + \text{CMD} + \text{\&c.}, < 4\text{R}.$$

XXX.*

Theorem. *In two polar polyhedrals, the face angles of the one are the supplements of the dihedrals formed by the corresponding planes of the other.*

For if, at a point, M, in the edge, EF, of two faces of a polyhedral, MA and MB be drawn perpendicular to those faces respectively, also MC and MD be drawn in the faces, perpendicular to the edge, these four perpendiculars will be in the same plane (2, Cor. 3). Also $\angle$ AMB will equal the face angle of a polar polyhedral corresponding to the dihedral $\overline{\text{CEFD}}$ of the given faces, which is measured by $\angle$ CMD (27, Def. 2).



Then $\angle \text{AMB} + \text{BMC} + \text{CMD} + \text{DMA} = 4\text{R},$
but $\angle \text{BMC} + \text{DMA} = 2\text{R}:$
hence $\angle \text{AMB} + \text{CMD} = 2\text{R}.$

XXXI.*

Theorem. *In any convex polyhedral of* n *faces, the dihedral angles formed by the adjacent faces are together less than* 2n*R*, *and greater than* 2n*R*—4*R*.

For the sum of the dihedrals of a polyhedral, plus the face angles of its polar polyhedral $= 2n$R (30): hence the sum of the dihedrals $< 2n$R. Also the sum of the face angles of the polar polyhedral < 4R (29): hence the sum of the dihedrals $> 2n$R—4R.

Scholium. Let S = sum of dihedrals of a polyhedral, and P = sum of face angles of its polar angle. Then $\text{S} + \text{P} = 2n\text{R}$ (30): hence $\text{S} < 2n\text{R}$. Also $\text{P} < 4\text{R}$ (29); hence $\text{S} > 2n\text{R}—4\text{R}$.

XXXII.

Theorem. *If two trihedrals have two face angles of the one*

equal to two face angles of the other, each to each, and the included dihedrals equal, the two trihedrals will be congruent, or symmetrical.

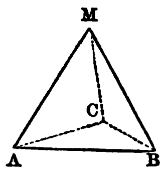
For the equal parts of one may be so applied to those of the other, or its symmetrical angle, that the third faces will correspond; and the angles will agree in all their parts.

XXXIII.

Theorem. *If two trihedrals have two dihedrals of the one equal to two dihedrals of the other, each to each, and the included face angles equal, the two trihedrals will be congruent, or symmetrical.* (Proof similar to the last.)

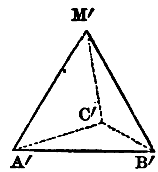
XXXIV.

Theorem. *If two trihedrals have three face angles of the one equal to three face angles of the other, each to each, their dihedrals will be equal each to each, and the trihedrals will be congruent, or symmetrical.*



HYPOTH. $AMB = A'M'B'$, $AMC = A'M'C'$, $CMB = C'M'B'$.

TO BE PROVED. The dihedral $A\overline{MC}B = A'\overline{M'C'}B'$, &c.



PROOF. Take $MC = M'C'$. Through C and C′ respectively, pass the planes $ACB \perp MC$, and $A'C'B' \perp M'C'$. The angles ACB and A′C′B′ measure the dihedrals $A\overline{MC}B$ and $A'\overline{M'C'}B'$.

Then $\triangle\ AMC \cong A'M'C'$ (I., 21),
also $\triangle\ BMC \cong B'M'C'$ (I., 21);
that is, $AM = A'M'$, $AC = A'C'$, $BM = B'M'$, and $BC = B'C'$:
hence $\triangle\ AMB \cong A'M'B'$ (I., 20), and $AB = A'B'$.
Then $\triangle\ ACB \cong A'C'B'$ (I., 24),
and $\angle\ ACB = A'C'B'$, or $\angle\ A\overline{MC}B = A'\overline{M'C'}B$;
and the trihedrals, M—ABC and M′—A′B′C′, are congruent, or symmetrical (32).

SCHOLIUM. If the angles AMC and BMC are obtuse, the perpendiculars CA and CB will not meet MA and MB. In that case, prolong CM through the vertex, M; and through any point, C, of this prolongation, pass a plane perpendicular to CM. It will cut the other two edges in two points, A and B. Apply the same construction to the angle M′—A′B′C′; and the demonstration remains the same. If AMC and BMC are right angles, the face angles, AMB and A′M′B′, measure the dihedrals, $A\overline{MC}B$ and $A'\overline{M'C'}B'$, and are equal by hypothesis.

XXXV.*

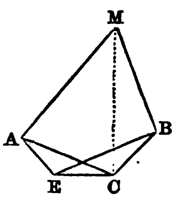
Theorem. Conversely, *if two trihedrals have three dihedrals of the one equal to three dihedrals of the other, each to each, their face angles will be equa leach to each, and the trihedrals will be congruent, or symmetrical.*

HYPOTH. Dihedral $A\overline{MC}B = A'\overline{M'C'}B'$, &c.

TO BE PROVED. Face angle AMB = A′M′B′, &c. (See figures in XXXIV.)

PROOF. The face angles of the polar trihedrals are equal each to each, being supplements of the equal dihedrals (30); hence the dihedrals of the polar trihedrals are also equal (34); and the face angles of the given trihedrals are equal, being supplements of these dihedrals (30).

XXXVI.



Theorem. *If two face angles of a trihedral are equal, the opposite dihedrals are equal.*

HYPOTH. ∠ AMC = BMC.

TO BE PROVED. Dihedral $A\overline{MB}C = B\overline{MA}C$.

PROOF. From C draw CE ⊥ plane AMB, EA ⊥ MA, EB ⊥ MB; join AC and BC; then ∠ EBC and EAC measure the dihedrals $A\overline{MB}C$ and $B\overline{MA}C$ (16).

Also $\triangle$ AMC $\cong$ BMC (I., 21):
hence CA $=$ CB, and $\triangle$ AEC $\cong$ BEC (I., 29, Cor.):
hence $\angle$ EBC $=$ EAC; that is, the dihedral $\overline{\text{AMB}}$C $=$ B$\overline{\text{MA}}$C.

XXXVII.

Theorem. Conversely, *if two dihedrals of a trihedral are equal, the opposite face angles are equal.*

HYPOTH. The dihedral $\overline{\text{AMB}}$C $=$ B$\overline{\text{MA}}$C; or, $\angle$ EBC $=$ EAC.

TO BE PROVED. Face angle AMC $=$ BMC.

PROOF. Like the last.

COR. 1. If the three face angles of a trihedral are equal, the three dihedrals are also equal (36).

COR. 2. Conversely, if the three dihedrals of a trihedral are equal, the three face angles are also equal.

BOOK VI.

POLYHEDRONS.

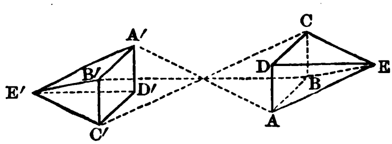
I.

Def. 1. A **geometrical solid** is a portion of space enclosed by surfaces. When the bounding surfaces are all planes, the solid is called a **polyhedron.** The planes are the **faces,** and their intersections the **edges,** of the polyhedron. A **diagonal** of a polyhedron is a line joining any two vertices not in the same plane.

Cor. The edges of a polyhedron are straight lines (V., 1, Cor. 2), and the faces are polygons.

Def. 2. Polyhedrons are **congruent,** that is, may be so applied to each other as to correspond in all their parts, when their edges and angles (plane and solid) are equal each to each, and arranged in the same order.

Def. 3. Two polyhedrons are **symmetrical** when the edges and angles are equal each to each, but arranged in the opposite order. If two equal faces, ABCD and A′B′C′D′, be compared with each other, so that ∠ A falls on A, B on B′, &c., it is evident that the solids will extend in opposite directions, and hence cannot be so compared with each other as to coincide.

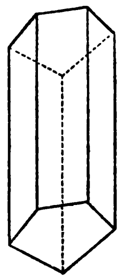


DEF. 4. Two polyhedrons are **similar** when the angles of the one are equal to the angles of the other, each to each, and the homologous edges proportional; that is, when the solid angles are congruent each to each, and the homologous faces are similar polygons.

THE PRISM.

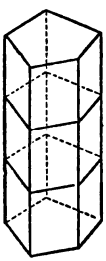
II.

DEF. 1. If, from the vertices of a polygon to a parallel plane, parallel lines be drawn, and a plane be passed through each side of the polygon, and the lines drawn from its extremities, the solid enclosed is called a **prism.**



DEF. 2. The two parallel polygons are called the **bases,** the other faces taken together the **lateral** or **convex surface,** the parallel lines the **lateral edges,** and the perpendicular distance between the bases the **altitude,** of the prism.

COR. 1. The lateral faces are parallelograms (V., 22), and the bases are congruent polygons. (Give proof.)

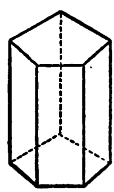


COR. 2. The lateral edges are all equal to each other.

COR. 3. If two parallel planes cut a prism, the sections are congruent polygons; for they form the bases of a new prism.

DEF. 3. A prism is **triangular, quadrangular, pentagonal,** &c., according as it has three, four, five, &c., lateral faces.

DEF. 4. A **right prism** is one whose lateral edges are perpendicular to the bases.



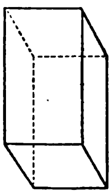
DEF. 5. An **oblique prism** is one whose lateral edges are oblique to the bases.

DEF. 6. A **regular prism** is a right prism whose bases are regular polygons.

III.

Def. 1. A **parallelopiped** is a prism whose bases are parallelograms.

Cor. 1. Any two opposite faces of a parallelopiped are parallel and congruent. They are parallel by V., 21, Cor., and congruent, since they are mutually equiangular (V., 5) and mutually equilateral, any two homologous sides being the opposite sides of the same parallelogram.

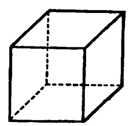


Cor. 2. A parallelopiped has six faces, any one of which may be taken as the base.

Def. 2. A **rectangular parallelopiped** is a right parallelopiped whose bases are rectangles. (II., Def. 4.)

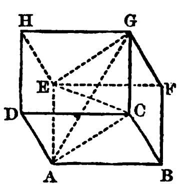
Cor. 3. All the faces of a rectangular parallelopiped are rectangles.

Def. 3. A **cube** is a rectangular parallelopiped, all of whose faces are squares.



IV.

Theorem. *The four diagonals of a parallelopiped bisect each other.*



Hypoth. ABCD—E is a parallelopiped.

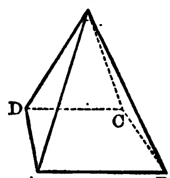
To be Proved. The diagonals AG, BH, CE, and DF, bisect each other.

Proof. Through the equal and parallel edges, AE and CG, pass a plane, cutting the parallel faces ABCD and EFGH in AC and EG. Then ACGE is a parallelogram, and the diagonals AG and CE bisect each other (I., 41). In like manner it may be shown that AG and BH, also AG and FD, bisect each other. Hence the four diagonals bisect each other. Construct figure, and show that AG bisects BH.

PYRAMIDS.

V.

Def. 1. If, from the vertices of any polygon, lines be drawn to a point not in the plane of the polygon, and planes be passed through the triangles thus formed, the enclosed space will be a **pyramid**; as, M—ABCD.



Def. 2. The polygon ABCD is called the **base**; the lines MA, MB, . . . the **lateral edges**; the triangles MAB, MBC, . . . taken together, the **lateral surface**; and the perpendicular let fall from the **vertex** M to the plane of the base, the **altitude**.

Def. 3. A pyramid is **triangular, quadrangular, pentagonal**, &c., according as its base has three, four, five, &c., sides.

Def. 4. A triangular pyramid has but four faces, and is called a **tetrahedron**.

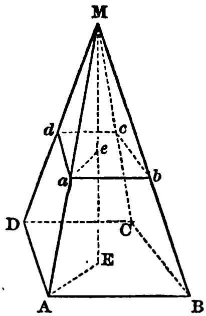
Def. 5. A pyramid is **regular** when its base is a regular polygon, and the perpendicular let fall from the vertex upon the base passes through the centre of the polygon. This perpendicular is called the **axis** of the regular pyramid; and a perpendicular drawn from the vertex to the sides of the base, the **slant height**.

Cor. Since a circle may be circumscribed about, and one inscribed within, the base (IV. 2), it follows (from V. 3) that the lateral edges of a regular pyramid are all equal; also the slant height is the same to all sides of the base.

VI.

Theorem. *If a pyramid be cut by a plane parallel to the base,*

1. *The edges and altitude will be divided proportionately.*
2. *The section will be a polygon similar to the base.*



Hypoth. The pyramid M—ABCD is cut by a plane, *abcd* ∥ ABCD; and ME is the altitude.

To be Proved. 1. Altitude ME : M*e* = MA : M*a* = MB : M*b*, &c.

Proof. Through MA and ME pass a plane. The intersection AE ∥ *ae* (V., 19); also AB ∥ *ab*, BC ∥ *bc*, &c.

Hence ME : M*e* = MA : M*a* = MB : M*b*, &c. (III., 17).

To be Proved. 2. Section *abcd* ∼ ABCD.

Proof. AB ∥ *ab*, BC ∥ *bc*, &c.

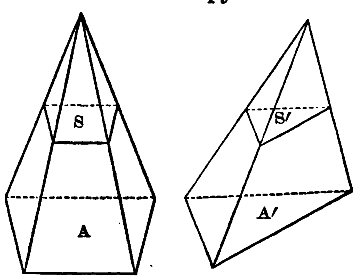
Hence ∠ ABC = *abc*, BCD = *bcd*, &c. (V., 5); that is, *abcd* and ABCD are mutually equiangular.

Also in the triangles MAB, MBC, &c., the line *ab* ∥ AB, *bc* ∥ BC, &c., and △ MAB ∼ M*ab*, MBC ∼ M*bc*, &c. (III., 21): hence MB : M*b* = AB : *ab* = BC : *bc*, &c.

In like manner, it may be shown that BC : *bc* = CD : *cd* = DA : *da*; that is, *abcd* and ABCD have their homologous sides proportional: hence they are similar.

Cor. 1. ABCD : *abcd* = $AB^2 : ab^2 = MA^2 : Ma^2 = ME^2 : Me^2$ (III., 30).

Cor. 2. If two pyramids with equal altitudes are cut by planes parallel to, and at equal distances from, the bases, the two sections will be proportional to the bases.



For if H is the altitude of two pyramids, A and A′ the areas of their bases respectively, S and S′ the areas of two sections parallel to the bases, and at the perpendicular distance D from the vertices, then

$$A : S = H^2 : D^2,$$

and $$A' : S' = H^2 : D^2 :$$

hence $$A : S = A' : S'.$$

COR. 3. If the bases A and A′ are equal, the sections S and S′, at equal distances from, and parallel to, the bases, are equal.

DEF. 1. A **truncated pyramid** is the portion of a pyramid included between the base and a plane cutting the pyramid. When the cutting plane is parallel to the base, the truncated pyramid is called a **frustum** of a pyramid.

DEF. 2. The **altitude** of a frustum is the perpendicular distance between its bases.

VII.

DEF. 1. A **regular polyhedron** is one whose solid angles are all equal to each other, and whose faces are all equal regular polygons.

DEF. 2. A polyhedron of four faces is called a **tetrahedron**; one of six faces, a **hexahedron**; one of eight faces, an **octahedron**; one of twelve faces, a **dodecahedron**; and one of twenty, an **icosahedron**.

Theorem. *There can be but five regular polyhedrons.*

PROOF. Each solid angle must be contained by the plane angles of three or more equal regular polygons; and the sum of these plane angles must be less than four right angles (V., 29).

1. If the faces are equiangular triangles, each plane angle $= \frac{2}{3}R$ (I., 17, Cor. 5). Each solid angle may be contained by three of these angles, forming the **tetrahedron**; by four, forming the **octahedron**; or by five, forming the **icosahedron**. No convex solid angle can be formed by a greater number; for six of the plane angles $= \frac{12}{3}R = 4R$.

2. If the faces are squares, each solid angle may be contained by three plane angles, forming the **hexahedron**,

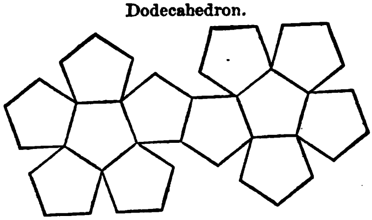
but not by four, or more, since they would be together equal to or greater than 4R.

3. If the faces are regular pentagons, each solid angle may be contained by three plane angles $= \frac{18}{5}$R (I., 35), forming the **dodecahedron.** Four plane angles $= \frac{24R}{5} > 4R$.

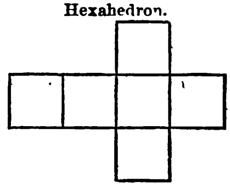
4. A convex solid angle cannot be formed by the angles of a regular hexagon; for three of these angles $= \frac{12}{3}R = 4R$. The same is true of the regular heptagon, octagon, &c.

SCHOLIUM. To construct models of the regular polyhedrons, draw the following diagrams on pasteboard; cut them out entire; at the dividing lines, cut the pasteboard half through;

Dodecahedron.

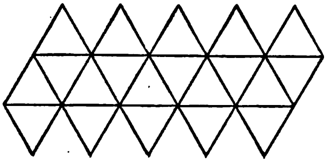


Hexahedron.

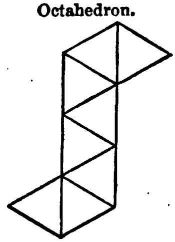


bend them into the form of the respective polygons; and glue the edges together:—

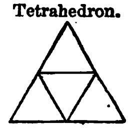
Icosahedron.



Octahedron.



Tetrahedron.



IX.*

Theorem.† *If any polyhedron has* f *faces*, v *vertices, and* e *edges, then* f + v = e + 2.

Proof. Take one face out of the polyhedron, and there will remain an open polyhedron with $f - 1$ faces, v vertices, and e edges. Take away from the border of the opening a face, and there will remain $f_1 = f - 2$ faces, $v_1 = v$ vertices, and $e_1 = e - 1$ edges. Remove a second face from the open polyhedron, and there will remain $f_2 = f_1 - 1$ faces, $v_2 = v_1 - m$ vertices, and $e_2 = e_1 - (m + 1)$. If all the faces be removed, one by one, the number of faces, vertices, and edges remaining, will be expressed by the following table, in which $m, m_1, m_2, \ldots$ is the number of isolated vertices, and $m + 1, m_1 + 1, m_2 + 1, \ldots$ the number of isolated edges on the face removed. When $f - 1$ faces are removed, the number m_n of edges and vertices remaining will be the same as the number of sides of the face remaining:—

	FACES.		FACES.	VERTICES.	EDGES.	
Remove	1,	and	$f-1$	v	e	remain.
"	2	"	$f_1 = f-2$	$v_1 = v$	$e_1 = e - 1$	"
"	3	"	$f_2 = f_1 - 1$	$v_2 = v_1 - m$	$e_2 = e_1 - (m + 1)$	"
"	4	"	$f_3 = f_2 - 1$	$v_3 = v_2 - m_1$	$e_3 = e_2 - (m_1 + 1)$	"
"	5	"	$f_4 = f_3 - 1$	$v_4 = v_3 - m_2$	$e_4 = e_3 - (m_2 + 1)$	"
.	.	.	. .	. . .	. . . .	.
"	$f-1$	"	1	m_n	m_n	"

Transposing the terms of the above equations (thus, $f = f_1 + 2$, $v = v_1$, $e = e_1 + 1$, &c.), and substituting in the expression $f - 1 + v - e$, we have $f - 1 + v - e = f_1 + 2 - 1 + v_1 - (e_1 + 1) = f_1 + v_1 - e_1 = f_2 + v_2 - e_2 = f_3 + v_3 + e_3 = \ldots 1 + m_n - m_n = 1$; that is,

$$f - 1 + v - e = 1, \text{ or } f + v = e + 2.$$

† The proof of this theorem was first published by Euler (1752), but has been found in a manuscript left by Descartes, who died 1650.

Exercise. How many edges has each of the five regular polyhedrons?

Cor. The number of face angles in any polyhedron is twice as great as the number of edges; for each face has the same number of sides and angles, and two sides form but one edge.

Hence the number of face angles in a polyhedron is even; the number of triangular, pentagonal, heptagonal, &c., faces, must be even; and the number of solid angles formed by 3, 5, 7, &c., plane angles, must be even.

XI.*

Theorem. *The sum of all the face angles of any polyhedron is equal to four right angles, taken as many times as the polyhedron has vertices, less two.*

To be Proved. The sum of the face angles $S = 4R (v - 2)$.

Proof. Let $A_1, A_2, \ldots$ be the sums of the angles, and $n_1, n_2, n_3, \ldots$ the number of sides, of the faces respectively; then $A_1 = 2Rn_1 - 4R$, $A_2 = 2Rn_2 - 4R$, $A_3 = 2Rn_3 - 4R$ (I., 35, Cor. 1). Adding these equations, member to member, we have $A_1 + A_2 + A_3 + \ldots = S = 2R (n_1 + n_2 + n_3 \ldots) - 4Rf = 2R (2e) - 4Rf$ (10, Cor.); or,

$$S = 4R (e - f).$$

$$\text{But} \quad e - f = v - 2 \ (10):$$

$$\text{hence} \quad S = 4R (v - 2).$$

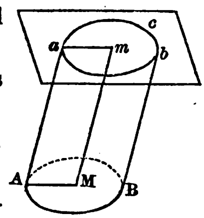
BOOK VII.

THE THREE ROUND BODIES.

THE CYLINDER.

I.

DEF. 1. The locus of the parallel lines drawn from all the points in the circumference of a circle to a plane parallel to that circle is a **cylindrical surface**; * and the solid enclosed is a **cylinder.**



DEF. 2. Any one of the parallel lines is called an **element**; as, Aa, Bb.

COR. 1. All the elements, Aa, Bb, . . . are equal (V., 22).

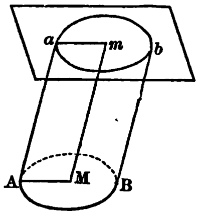
COR. 2. The locus of all the intersections of the elements and the plane is the circumference of a circle equal to the given circle. For from the centre, M, draw Mm || Aa; then Mm = Aa, and AMma is a parallelogram: hence MA = ma. For the same reason, the distance from m to the extremity, b, of any element, Bb, equals the radius MB: hence abc is a circle drawn from m as a centre, with a radius equal to AM.

DEF. 3. The two circles are called the **bases** of the cylinder; the line Mm, joining their centres, the **axis**; the perpendicular distance between the bases, the **altitude**; and the locus of the elements, the **lateral surface.**

COR. 3. *Every section of a cylinder made by a plane paral-*

* **A cylindrical surface in its most extended sense is the locus of the parallel lines drawn through all the points of any curve.**

lel to the bases is a circle equal to the bases; for it is the base of a new cylinder cut off by this plane.



Cor. 4. *Every section of a cylinder made by a plane passing through an element is a parallelogram;* for a plane passing through an element, Aa, will cut the circumference of the base in a second point, B. Through B draw Bb || Aa; then Bb is in the plane AaB, and, by the definition, is an element of the cylinder: hence Bb = Aa, and AabB is a parallelogram (I., 39).

Def. 4. If the plane AabB be turned about Aa, the intersection Bb will at length fall upon Aa. In this position, the plane has but one line common with the cylinder, and is called a **tangent plane.**

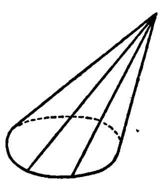
Def. 5. A **right cylinder** is one whose elements are perpendicular to its base. It may be generated by the revolution of a rectangle about one of its sides. This side is the **axis** of the cylinder. The opposite side generates the curved surface; and the other two sides, the bases.

Scholium. A cylinder may be considered a prism of an infinite number of sides (IV., 10).

THE CONE.

II.

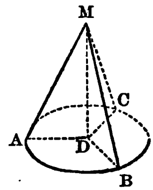
Def. 1. The locus of all the lines drawn through a point, and the circumference of a circle not in the same plane, is a **conical surface**; and the solid enclosed is a **cone.**



Def. 2. The lines are called the **elements** of the cone; the point, the **vertex;** the circle, the **base**; the perpendicular distance of the vertex from the base, the **altitude**; the line drawn from the vertex to

the middle of the base, the **axis**; and the locus of the elements, the **lateral surface.**

DEF. 3. A **right cone** is a cone whose axis is perpendicular to its base. It may be generated by the revolution of a right-angled triangle, ADM, about one of the perpendicular sides, MD.

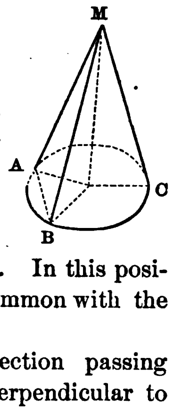


SCHOLIUM. A cone may be considered a pyramid of an infinite number of sides (IV., 10).

III.

Theorem. *Every section of a cone made by a plane passing through the vertex is a triangle.*

For let a plane pass through the vertex, M, cutting the base in AB. Join MA and MB; then MA and MB are elements of the cone, and lie in the plane, since they have each two points in common with the plane: hence they are the intersection of the plane and lateral surface; and MAB is a triangle.



DEF. If the plane MAB be turned about MA, the intersection, MB, will fall upon MA. In this position, the plane has but one line, MA, in common with the cone; and is called a **tangent plane.**

EXERCISE. Prove that the triangular section passing through the axis of an oblique cone, and perpendicular to the base, has the same altitude as the cone, also the greatest and least elements for its sides.

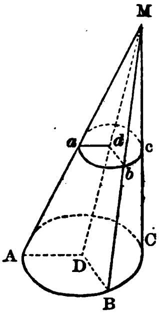
IV.

Theorem. *Every section of a cone made by a plane parallel to the base is a circle.*

HYPOTH. In the cone M—ABC, the plane section, *abc*, is parallel to the base ABC.

TO BE PROVED. *abc* is a circle.

Proof. Let MD be the axis, cutting abc in d. Through MD, and any element, MA, MB, . . . pass planes intersecting ABC in AD, BD, . . . and abc in ad, bd, . . .



Then $ad \parallel$ AD, $bd \parallel$ BD (V., 19), and $\frac{AD}{ad} = \frac{MD}{Md}$, $\frac{BD}{bd} = \frac{MD}{Md}$ . . . (III., 21) : hence

$$\frac{AD}{ad} = \frac{BD}{bd} = \ldots;$$

but $\quad AD = BD = \ldots;$

hence $\quad ad = bd = \ldots;$ that is, abc is a circle described from d as a centre with a radius ad.

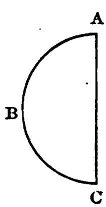
Cor. The axis of a cone passes through the centre of all the circular sections parallel to the base.

Def. A **truncated cone** is the portion of a cone included between the base and a plane cutting the cone. When the cutting plane is parallel to the base, the truncated cone is called a **frustum of a cone**. The **altitude** of a frustum is the perpendicular distance between its bases.

THE SPHERE.†

V.

Def. The locus of all the points that are equally distant from a given point is a **spherical surface**; and the solid enclosed by that surface is a **sphere**. The given point is the **centre** of the sphere; the line drawn from the centre to the surface, the **radius**; and the line drawn through the centre, and terminated both ways, the **diameter**.



A sphere may be generated by the revolution of a semicircle, ABC, about its diameter, AC.

† The recitation-room should be furnished with a spherical blackboard, on which the student should draw the diagrams of spherical surfaces.

Cor. 1. All the radii of a sphere are equal, and all the diameters are equal; each being double the radius.

Cor. 2. A point is without or within the sphere, according as its distance from the centre is greater or less than the radius.

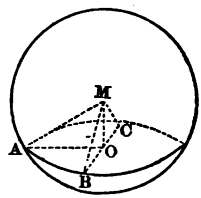
Cor. 3. Spheres of equal radii or of equal diameters are congruent.

VI.

Theorem. *Every section of a sphere made by a plane is a circle.*

Hypoth. ABC is a plane section of the sphere, whose centre is M.

To be Proved. ABC is a circle.



Proof. Draw MA, MB, MC, from the centre to any points, A, B, C, of the intersection. Draw MO $\perp$ the plane ABC, and join OA, OB, OC; then MOA, MOB, MOC, are right-angled triangles, having the hypothenuse $MA = MB = MC$, and the perpendicular MO common. Hence $OA = OB = OC$ (I., 29, Cor.); and ABC is a circle, whose centre is O.

Def. 1. If a plane passes through the centre of a sphere, the section is called a **great circle**. A section made by a plane which does not pass through the centre is called a **small circle**.

Cor. 1. The radius of a great circle is equal to the radius of the sphere; and the radius of a small circle is less than the radius of the sphere.

Cor. 2. All great circles on the same sphere, or equal spheres, are equal; also small circles at equal distances from the centre of the sphere are equal; and, conversely, two equal small circles are at equal distance from the centre of the sphere.

Cor. 3. On the same sphere, or equal spheres of two

unequal small circles, the less is more remote from the centre, and conversely.

SCHOLIUM. The above corollaries follow from the equation, $R^2 = r^2 + p^2$, in which $R = MA$, the radius of the sphere, $r = OA$, the radius of the section, and $p = MO$, the perpendicular. Thus, for a great circle, $p = 0$, and the equation gives $R^2 = r^2$; or, $R = r$: hence Cor. 1.

COR. 4. A perpendicular drawn from the centre of a sphere to a small circle passes through the centre of that circle.

COR. 5. Conversely, a perpendicular erected at the centre of a small circle passes through the centre of the sphere, and, when prolonged in both directions to the surface, is a diameter of the sphere.

DEF. 2. The extremities of a diameter perpendicular to a circle on the sphere are called the **poles**, and the diameter the **axis**, of that circle.

COR. 6. All the parallel circles on a sphere have the same axis and poles.

COR. 7. Every great circle divides a sphere and its surface into two congruent parts; for if the two parts be separated, and so applied to each other that their bases coincide, and their convexities turn in the same direction, their surfaces will coincide: otherwise there would be points in the spherical surface unequally distant from the centre.

COR. 8. Any two great circles bisect each other; for they have the same centre, and their common section is a diameter of both.

COR. 9. An arc of a great circle may be drawn through any two points on a surface of a sphere; for a plane may be passed through these points and the centre of the sphere (V., 1), and its section will be a great circle. If the two points are the extremities of a diameter, they are in the same straight line with the centre; and an infinite number of great circles may be passed through them.

COR. 10. An arc of a circle may be passed through any three points on the surface of a sphere.

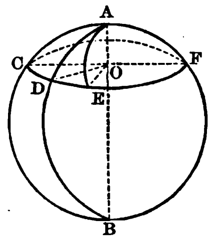
VII.

Theorem. *All the points in the circumference of a circle on a sphere are equally distant from each of its poles.*

HYPOTH. CDEF is a circle on a sphere, AB its axis, and A a pole.

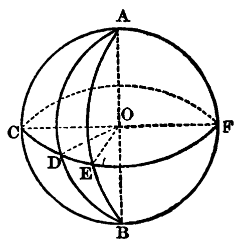
TO BE PROVED. AC = AD = AE =, &c.

PROOF. Since the axis, AB, passes through the centre of the circle, CDEF, at right angles, the triangles AOC, AOD, AOE, . . . are congruent. Hence the side AC = AD = AE, . . . &c.



COR. 1. All the arcs of great circles, AC, AD, AE, &c., drawn from a pole of a circle to points in its circumference, are equal, since their chords are equal (II., 6).

COR. 2. The distance of the circumference of a great circle from its pole is a *quadrant*, or *quarter of a great circle*; for if EDCF is a great circle, and A its pole, the arcs of great circles, AC, AD, AE, . . . have the same measure as the right angles, AOC, AOD, AOE, . . . at their centre O (II., 9, Scholium).

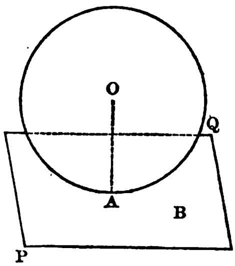


SCHOLIUM. By revolving the arc AC about the pole A, the point C will describe the circumference of a circle on the sphere. To draw an arc of a great circle through two points, C and D, first draw from C and D, as poles, arcs of great circles cutting each other in A. Then A is the pole of the arc required.

DEF. A plane is **tangent** to a sphere when it has but one point common with the sphere.

VIII.

Theorem. *A plane perpendicular to a radius at its extremity is tangent to the sphere.*



HYPOTH. The plane PQ ⊥ OA, the radius of a sphere.

TO BE PROVED. PQ is tangent to the sphere.

PROOF. Any other point, B, in the plane PQ, is farther from the centre, O, than the perpendicular distance OA, and is without the sphere: hence PQ is tangent to the sphere at the point A.

COR. 1. The radius drawn to the point of contact of a tangent plane is perpendicular to that plane.

COR. 2. The perpendicular drawn from the centre of a sphere to a tangent plane passes through the point of contact.

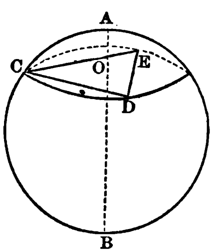
COR. 3. The perpendicular erected at the point of contact of a tangent plane passes through the centre of the sphere.

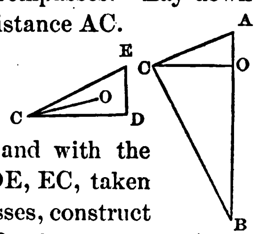
(Give indirect demonstrations of the above corollaries.)

IX.

Problem. *To find the diameter of a given sphere.*

SOLUTION. From any point, A, on the surface of the sphere, as a pole, describe a circumference, CDE, with the compasses. Lay down the rectilinear distance AC.



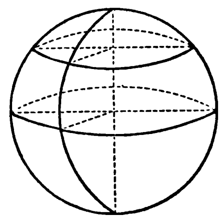


Take any three points, C, D, and E, in this circumference, and with the distances CD, DE, EC, taken with the compasses, construct the triangle CDE (II., 29). Find O, the centre of the circumscribed circle (II., 4), and with the distances OC and AC construct the right-angled triangle ACO. At C con-

struct the right angle ACB. Prolong AO to meet CB in B. Then it is evident, from what precedes, that AB is the diameter of the sphere.

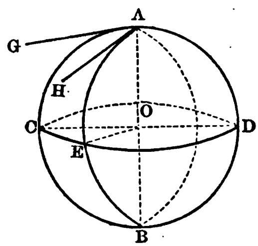
X.

Def. 1. The spherical surface included between two parallel planes cutting a sphere is called a **zone**; and the solid included is called a **spherical segment.** One of the planes may be tangent to the sphere.



Def. 2. If two great circles cut each other, they divide the surface of the sphere into four parts, each of which is called a **lune**; and the sphere into four parts, each of which is called a **spherical wedge,** or **ungula.** The lune, ACBE, is the base of the ungula.

Def. 3. A **spherical angle** is an angle formed by two arcs of great circles meeting at a point, and is the same as the angle of their tangents at that point; thus, $\angle$ CAE = GAH, where GA and HA are tangents of the arcs CA and EA.



XI.

Cor. 1. Since the tangents GA and HA lie respectively in the planes of the arcs CA and EA, and are perpendicular to their intersection, AB, it follows that $\angle$ CAE = GAH = C$\overline{AO}$E, the dihedral of the planes ACO and AEO.

If O is the centre of the sphere, and CED a great circle, whose pole is A, the angle COE and the arc CE have the same measure as the dihedral C$\overline{AO}$E: hence $\angle$ CAE = GAH = C$\overline{AO}$E = COE = arc CE; that is, *the angle of two arcs of great circles has the same measure as the dihedral of their planes, or as the arc of a great circle described from its ver-*

tex as a pole, and included between its sides (produced, if necessary).

Cor. 2. If two arcs of great circles cut each other, their vertical angles are equal.

Cor. 3. If A is the pole of a great circle, CED, and AC is an arc of a great circle, then $\angle$ ACE $=$ R; for diameter AOB $\perp$ plane CED (V., 1, Def. 2): hence plane ACO $\perp$ CED (V., 14), conversely, if CA $\perp$ CD, each arc passes through the pole of the other.

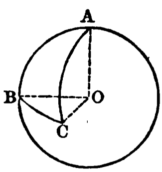
SPHERICAL POLYGONS.

XII.

Def. 1. A **spherical polygon** is a portion of the surface of a sphere bounded by three or more arcs of great circles, each of which is less than a semi-circumference.

Def. 2. A **spherical triangle** is a spherical polygon of three sides. It is called **isosceles, right-angled**, or **equilateral**, in the same cases as a plane triangle.

Def. 3. The planes of a great circle pass through the centre of the sphere: hence the planes of the sides of a spherical polygon form a polyhedral at the centre of the sphere. The portion of the sphere enclosed is called a **spherical pyramid**; as, O — ABC; the edges, OA, OB, OC, are the radii of the sphere; the angles, AOB, BOC, &c., are the face angles of the pyramid.



Cor. 1. The face angles, AOB, BOC, &c., have the same measure as the sides of the polygon, AB, BC, &c.; and the dihedrals of the faces equal the angles of the polygon: thus dihedral $A\overline{BO}C = \angle$ ABC. Hence *from any relation that has been shown to exist between the face angles and dihedrals of a polyhedral, we may infer the same relation between the sides and angles of a spherical polygon, and conversely.*

Cor. 2. The plane angles of a polyhedral are each less than

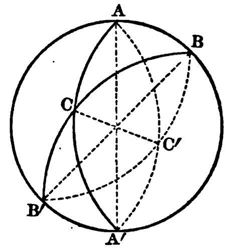
two right angles: hence each side of a spherical polygon is less than a semi-circumference (Def. 1).

XIII.

Theorem. *If three great circles intersect each other on the surface of a sphere, the sides and angles of the eight triangles thus formed are respectively equal, or supplements of each other.*

Hypoth. ABB′, ACC′, and BCC′ are three great circles.

To be Proved. The sides and angles of the triangles ABC, AB′C, . . . are equal each to each, or supplements of each other.



Proof. Since two great circles bisect each other (6, Cor. 8), the arcs BAB′ and BCB′ are semicircles: hence BA and AB′ are supplements of each other, and BC is the supplement of CB′.

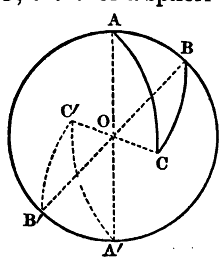
Also ∠ BAC is the supplement of ∠ CAB′, ∠ ABC = AB′C (11, Cor. 1).

In like manner it may be shown that the sides and angles of any other two triangles are equal each to each, or supplements of each other.

SYMMETRICAL SPHERICAL POLYGONS.

XIV.

Def. 1. If through the vertices, A, B, C, . . . of a spherical polygon, the diameters, AA′, BB′, CC′, . . . be drawn, and arcs of great circles, A′B′, B′C′, . . . be constructed, then ABC . . . and A′B′C′ . . . are **symmetrical polygons.**



Cor. 1. The sides and angles of two symmetrical polygons are equal each to each. For two sides, AB and A′B′, have

the same measure as the equal vertical angles, AOB and A′OB′ (12, Cor. 1), also $\angle$ ABC = A′B′C′, each being equal to the dihedral of the same two planes, that is, the planes AA′BB′ and BB′CC′.

Def. 2. The two solids, O — ABC . . . and O — A′B′C′ . . . are **symmetrical spherical pyramids.**

Cor. 2. The solid angles at their vertex O are symmetrical polyhedrals (V., 27, Def. 1), whose face angles and dihedrals respectively have the same measure as the sides and angles of the symmetrical polygons ABC . . . and A′B′C′ . . . (12, Cor.,).

Hence, *from any relation that has been shown to exist between the face angles and dihedrals of two symmetrical polyhedrals, we may infer the same relation between the sides and angles of two symmetrical spherical polygons, and conversely.*

Cor. 3. Two symmetrical polygons, ABC and A′B′C′, have their equal parts arranged in inverse order, and cannot be compared with each other so as to coincide (see Cor. 4).

Def. 3. Two polygons are also called symmetrical when their sides are equal, each to each, and arranged in inverse order, whatever be their position upon the sphere.

Cor. 4. Two symmetrical isosceles triangles may be so applied to each other as to coincide; for if ABC and A′B′C′ have their sides, AC = BC, and B′C′ = A′C′, the vertex B′ may be placed on A, and A′ on B: then B′C′ will coincide with its equal AC, and A′C′ with BC; and the triangles will be congruent.

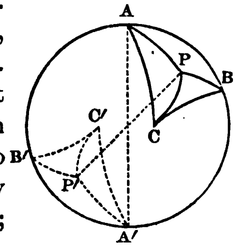
XV.

Theorem. *Two symmetrical spherical triangles are equal.*

Hypoth. ABC and A′B′C′ are two symmetrical spherical triangles, so situated that AA′, BB′, and CC′ are diameters of the sphere.

To be Proved. The triangle ABC = A′B′C′.

Proof. Let P be the pole of the small circle passed through A, B, and C (6, Cor. 10). Through P draw the arcs of great circles, PA, PB, PC, and the diameter PP′. Also through P′ draw the arcs of great circles, P′A′, P′B′, and P′C′. Then PA = PB = PC (7, Cor. 1); also P′A′ = P′B′ = P′C′, being respectively equal to PA, PB, and PC (14, Cor. 1); also AB = A′B′, BC = B′C′, CA = C′A′. Then PAB and P′A′B′ are symmetrical isosceles triangles: hence

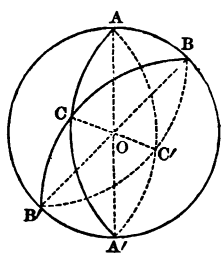


$$\text{PAB} \cong \text{P}'\text{A}'\text{B}' \text{ (14, Cor. 4)};$$
$$\text{also PBC} \cong \text{P}'\text{B}'\text{C}',$$
$$\text{and PCA} \cong \text{P}'\text{C}'\text{A}'.$$

Adding these equations, member to member, we have ABC = A′B′C′.

If the point P should fall without the triangle ABC, two of the equations should be added, and the third subtracted from their sum (III., 9).

Cor. 1. To each of the equal symmetrical triangles, A′B′C and ABC′, add the triangle ABC, and we have A′B′C + ABC = the lune ACBC′. Hence, *if two great circles intersect on the surface of a hemisphere, the sum of the opposite triangles thus formed is equal to a lune whose angle is equal to that formed by the circles.*

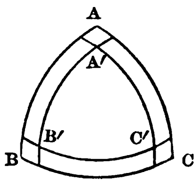


Cor. 2. By a demonstration similar to that in the theorem, it may be shown that the symmetrical spherical pyramids, O—ABC and O—A′B′C′, are equal: hence *two symmetrical spherical pyramids are equal.*

POLAR TRIANGLES.

XVI.

DEF. If from the vertices of a spherical triangle (polygon), ABC, as poles, arcs of great circles are described, these arcs form a second triangle, A′B′C′, which is called a **polar triangle (polygon).**



Theorem. *The two trihedrals (polyhedrals) formed by planes passing through the arcs of polar triangles (polygons) will be polar trihedrals (polyhedrals), whose vertices are at the centre of the sphere.*

For, since A is the pole of the arc B′C′, the distances AB′ and AC′ are quadrants: hence ∠ AOB′ and AOC′ are right angles, and AO is perpendicular to the plane B′OC′, being perpendicular to two lines in the plane (V., 2). Also BO ⊥ A′OC′, and CO ⊥ B′OA′: hence O — ABC and O — A′B′C′ are polar trihedrals (V., 27, Def. 2). The same is true of polygons of any number of sides.

COR. 1. Hence, *from any relation that has been shown to exist between the face angles and dihedrals of two polar trihedrals (polyhedrals), we may infer the same relation between the sides and angles of two polar triangles (polygons), and conversely.*

COR. 2. Since the distances BA′ and CA′ are quadrants, it follows that A′ is the pole of BC; also B′ is the pole of AC, and C′ is the pole of AB.

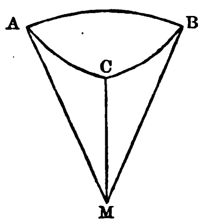
Hence ABC is the polar triangle of A′B′C′; and *every spherical triangle is the polar triangle of its polar triangle.*

EXERCISE. When is a polar triangle within the given triangle? when without? when partly within, and partly without? When does it coincide with it?

The student should illustrate all the following theorems by diagrams.

It has been shown, that

The sum of any two face angles of a trihedral is greater than the third (V., 28) ; thus AMC + CMB > AMB.



The sum of the face angles of any convex polyhedral is less than four right angles (V., 29).

In two polar polyhedrals, the face angles of the one are the supplements of the dihedrals formed by the corresponding planes of the other (V., 30).

XVII.

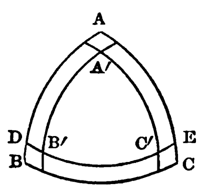
Hence (12, Cor. 1; 16, Cor.),
The sum of any two sides of a spherical triangle is greater than the third; thus AC + CB > AB.

XVIII.

The sum of the sides of any convex spherical polygon is less than the circumference of a great circle.

XIX.

In two polar spherical polygons, the sides of the one are the supplements of the angles of the other; thus $\angle$ A + arc B′C′ = 180°.



The following is a direct proof of the preceding theorem from the triangle: The arc DE $= \angle$ A (11, Cor. 1). DC′ $= 90°$, B′E $= 90°$. Hence, DC′ + B′E = DE + B′C′ $= 180°$, and $\angle$ A + B′C′ $= 180°$.

XX.

In any convex polyhedral of n *faces, the dihedrals formed by the adjacent faces are together greater than* 2nR − 4R, *and less than* 2nR (V., 31).

In any convex spherical polygon of n *sides, the sum of the angles are together greater than* 2nR − 4R, *and less than* 2nR.

XXI.

DEF. 1. The excess of the sum of the angles of a spherical polygon over $2n$R — 4R is called its **spherical excess.** If A, B, and C are the angles of a spherical triangle, its spherical excess $= \angle$ A + B + C — 180°; and generally the spherical excess of a polygon of n sides is $\angle$ A + B + + C + D . . —$(n - 2)180°$.

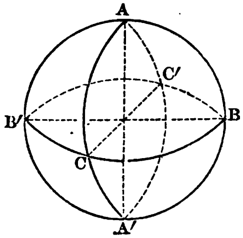
COR. 1. The spherical excess of a polygon cannot be greater than 4R.

DEF. 2. If a spherical triangle has two right angles, it is called a **bi-rectangular triangle.**

COR. 2. If ABC is a bi-rectangular triangle, the sides AB and AC, opposite the right angles C and B, are quadrants. (Give proof in full.) (V., 15, Cor. 3.)

DEF. 3. If a spherical triangle has three right angles, it is called a **tri-rectangular triangle.**

COR. 3. Each side of a tri-rectangular triangle is a quadrant (7, Cor. 2).



COR. 4. Three planes of great circles, each perpendicular to the other two, divide the surface of a sphere into eight tri-rectangular triangles. If S = surface of the sphere, and T = the tri-rectangular triangle, then

$$T = \frac{S}{8}; \text{ or, } S = 8T.$$

If two trihedrals have two face angles of the one equal to two face angles of the other, each to each, and the included dihedrals equal, the two trihedrals will be congruent or symmetrical (V., 32).

If two trihedrals have two dihedrals of the one equal to two dihedrals of the other, each to each, and the included face angles equal, the two trihedrals will be congruent or symmetrical (V., 33).

If two trihedrals have three face angles of the one equal to three face angles of the other, each to each, their dihedrals will be equal each to each; and the trihedrals will be congruent or symmetrical (V., 34).

Conversely, *if two trihedrals have three dihedrals of the one*

XXII.

If two spherical triangles on the same sphere, or equal spheres, have two sides and the included angle of the one equal to two sides and the included angle of the other, each to each, the two triangles will be congruent or symmetrical, and hence equal (15).

XXIII.

If two spherical triangles on the same sphere, or equal spheres, have two angles of the one equal to two angles of the other, each to each, and the included sides equal, the triangles will be congruent or symmetrical, and equal (15).

XXIV.

If two spherical triangles on the same sphere, or equal spheres, have three sides of the one equal to three sides of the other, each to each, their angles will be equal each to each; and the triangles are congruent or symmetrical, and equal (15).

XXV.

Conversely, *if two spherical triangles on the same sphere,*

equal to three dihedrals of the other, each to each, their face angles will be equal each to each; and the trihedrals will be congruent or symmetrical (V., 35).

If two face angles of a trihedral are equal, the opposite dihedrals are equal (V., 36).

If two dihedrals of a trihedral are equal, the opposite face angles are equal (V., 37).

Cor. 1. *If the three face angles of a trihedral are equal, the three dihedrals are equal.*

Cor. 2. Conversely, *if the three dihedrals of a trihedral are equal, the three face angles are equal.*

or equal spheres, have three angles of the one equal to three angles of the other, each to each, their sides will be equal each to each; and the triangles will be congruent or symmetrical, and equal (15).

XXVI.

If two sides of a spherical triangle are equal, the opposite angles are equal.

XXVII.

If two angles of a spherical triangle are equal, the opposite sides are equal; and the triangles are isosceles.

XXVIII.

If a spherical triangle is equilateral, it is equiangular;

and, conversely, *if a spherical triangle is equianguldr, it is equilateral.*

BOOK VIII.

MEASUREMENT OF SOLIDS.

PRISMS.

I.

Def. A plane section of a prism perpendicular to the lateral edges is called a **right section.**

A cylinder may be considered a prism with an infinite number of lateral edges (IV., 10).

Define *oblique*, *right*, and *rectangular prisms* (VI., 2).

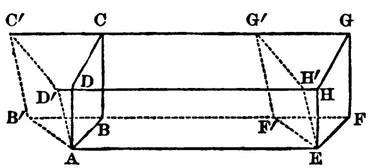
II.

Theorem. *Two prisms (cylinders) are equal when their lateral edges are equal, and right sections congruent.*

Hypoth. The prisms ABCD–G and AB′C′D′–G′ have their lateral edges, AE, DH, D′H′, . . . equal, and right sections congruent.

To be Proved. ABCD–G = AB′C′D′–G′.

Proof. The prisms may be so applied to each other, that two equal edges, AE, containing homologous vertices of the right sections, will coincide, and also the sections themselves. The other edges will respectively lie in the same straight line.

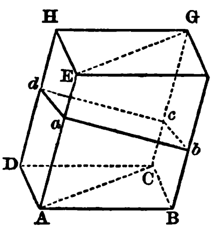


Then, since DH = D′H′, CG = C′G′, &c., it follows that DD′ = HH′, CC′ = GG′, &c.

Also the base ABCD ≅ EFGH (VI., 2, Cor. 1).

Hence, if the solid ABCD–C′ be applied to EFGH–G′, so that the base ABCD shall coincide with EFGH, the point D′ will fall on H′, C′ on G′, B′ on F′, and the solids will coincide throughout; that is, EFGH–G′ = ABCD–C′. To each member of this equation add ABCD–G′, and we have ABCD–G = AB′C′D′–G′.

Cor. 1. The plane ACGE, passed through two diagonally opposite edges, AE and CG, of a parallelopiped, divides it into two equal triangular prisms; for they have equal lateral edges, and the right sections, *abc* and *adc*, are congruent, each being half the parallelogram *abcd* (I., 37).



Cor. 2. An oblique prism is equal to a right prism which has an equal lateral edge, and a base congruent with a right section of the oblique prism.

III.

Theorem. *If two parallelopipeds have equal bases and equal altitudes, they are equal.*

Hypoth. ABCD–*a* and A′B′C′D′–*a*′ have the base ABCD = A′B′C′D′, and the same altitude.

To be Proved. ABCD–*a* = A′B′C′D′–*a*′.

Proof. Let the solids be so placed, that A′B′ and D′C′ produced will cut AD and BC produced, making HG = C′D′.

Then EFGH = ABCD, and HE = DA.

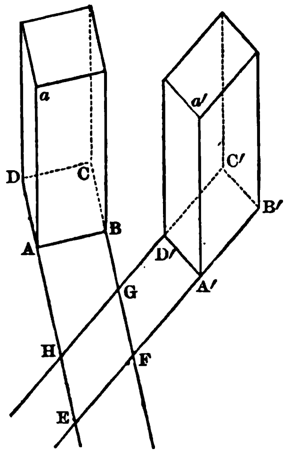
For EFGH = A′B′C′D′, having equal bases, HG and D′C′, and equal altitudes (III., 2): hence

EFGH = ABCD.

But EFGH and ABCD have the same altitudes, since they lie between the same parallels, DE and CF: hence

the base HE = DA. (III., 3.)

Upon EFGH suppose a parallelopiped, P, constructed, with an altitude equal to that of the given parallelopipeds.



Then, since DA = HE, and the right sections perpendicular to these edges are congruent, we have ABCD–a = P (2).

Also, since D′C′ = HG, and the right sections perpendicular to these edges are congruent, we have A′B′C′D′–a' = P (2) : hence

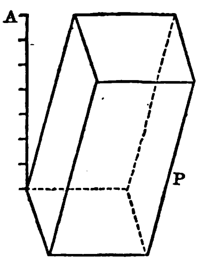
ABCD–a = A′B′C′D′–a'.

Cor. 1. A triangular prism is half a parallelopiped which has the same altitude and a double base (2, Cor. 1) : hence, *if two triangular prisms have equal bases and equal altitudes, they are equal.*

Cor. 2. Any parallelopiped is equal to a rectangular parallelopiped having an equal base and the same altitude.

IV.

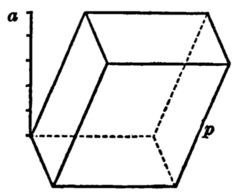
Theorem. *Two parallelopipeds having equal bases are to each other as their altitudes.*



Hypoth. P and p, are two parallelopipeds having equal bases, and the altitudes A and a.

To be Proved. P : p = A : a.

Proof. 1. The altitudes A and a are in the ratio of two whole numbers; for example, as 7 to 5. Divide A into 7 equal parts, and a will contain 5 of these parts : then



$$\frac{A}{a} = \frac{7}{5}.$$

Through the several points of division of A and a pass planes parallel to the bases. They will divide P into 7, and p into 5 parallelopipeds, all of which are equal (3).

Then $$\frac{P}{p} = \frac{7}{5}:$$

hence $$\frac{P}{p} = \frac{A}{a};\ \text{or, } P : p = A : a.$$

2. If the altitudes are incommensurable, divide a into n equal parts: then A will contain m of those parts and a small fraction.

Hence $$\frac{A}{a} = \frac{m}{n} \text{ to within } \frac{1}{n},$$

and $$\frac{P}{p} = \frac{m}{n} \text{ to within } \frac{1}{n},$$

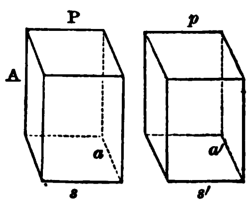
however far the approximation be carried.

Hence $$\frac{P}{p} = \frac{A}{a}\ (\text{II., } 8);$$

or, $$P : p = A : a.$$

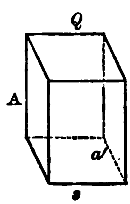
V.

Theorem. *Two rectangular parallelopipeds having equal altitudes are to each other as their bases.*



HYPOTH. P and p are two rectangular parallelopipeds, having the same altitude A, and the bases $s \times a$ and $s' \times a'$ respectively.

TO BE PROVED. $P : p = s \times a : s' \times a'$.



PROOF. Construct a rectangular parallelopiped, Q, with an altitude A, and base whose dimensions are s and a'.

Then $\frac{P}{Q} = \frac{a}{a'}$, and $\frac{Q}{p} = \frac{s}{s'}$ (4): since $s \times A$ and $s' \times A$ may be considered the bases, a and a' the altitudes, of P and p.

Hence $$\frac{P}{p} = \frac{s \times a}{s' \times a'};$$

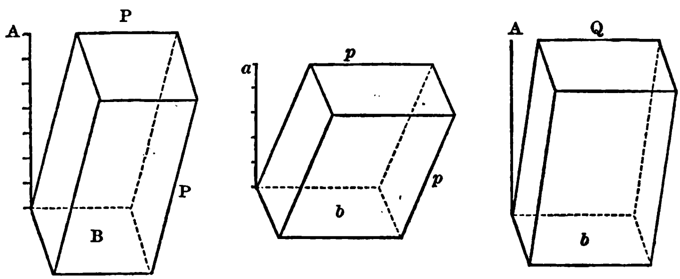
or, $$P : p = s \times a : s' \times a'.$$

Cor. Since any parallelopiped is equal to a rectangular parallelopiped having an equal base and the same altitude (3, Cor. 2), it follows that *any two parallelopipeds having equal altitudes are to each other as their bases.*

VI.

Theorem. *Any two parallelopipeds are to each other as the products of their bases by their altitudes.*

Hypoth. P and p are two parallelopipeds whose bases are B and b, and whose altitudes are A and a respectively.



To be Proved. $P : p = A \times B : a \times b.$

Proof. Construct a parallelopiped, Q with a base b, and altitude A.

Then $$\frac{P}{Q} = \frac{B}{b} \text{ (5, Cor.)},$$

and $$\frac{Q}{p} = \frac{A}{a} \text{ (4)}:$$

hence $$\frac{P}{p} = \frac{A \times B}{a \times b};$$

or, $$P : p = A \times B : a \times b.$$

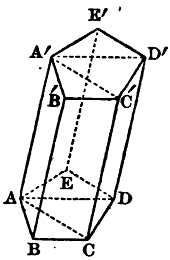
COR. If $a = 1$, and $b = 1$, the solid p will be a *unit of volume.*

Then $$P : 1 = A \times B : 1;$$
or, $$P = A \times B;$$
that is, *the volume of any parallelopiped is equal to the product of its base by its altitude.*

VII.

Theorem. *The volume of any prism (cylinder) is equal to the product of its base by its altitude.*

PROOF. A triangular prism is half a parallelopiped having a double base and the same altitude (2, Cor. 1): hence the volume of a triangular prism is equal to the product of its base by its altitude. (6 Cor.)



Any prism, ABCDE–A′, may be divided into triangular prisms by passing planes through an edge, AA′, and the diagonals AD, AC, of the base.

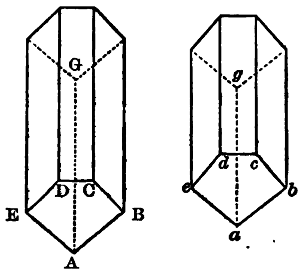
Then, if A is the altitude of the prism, we have,
$$ABC\text{–}A' = ABC \times A,$$
$$ACD\text{–}A' = ACD \times A,$$
$$ADE\text{–}A' = ADE \times A:$$
hence $$ABCDE\text{–}A' = ABCDE \times A;$$
that is, a prism, ABCDE–A′, is equal to the product of its base by its altitude.

COR. 1. *Prisms having equal bases are to each other as their altitudes; prisms having equal altitudes are to each other as their bases; and any two prisms are to each other as the products of their bases by their altitudes.*

COR. 2. An oblique prism is equal to a right prism whose base is congruent with its right section, and altitude equal to its lateral edges (2, Cor. 2): hence *the volume of any prism (cylinder) is equal to the product of a right section by a lateral edge.*

VIII.

Theorem. *Similar prisms are to each other as the cubes of their homologous edges.*



HYPOTH. The prism ABCDE–G $\sim$ *abcde–g*.

TO BE PROVED. ABCDE–G : *abcde–g* $= AB^3 : ab^3 = AG^3 : ag^3$.

PROOF. Let B and *b* denote the bases ABCDE and *abcde*, H and *h*, the altitudes, of the prisms.

Then ABCDE–G : *abcde–g* $= B \times H : b \times h$ (7, Cor. 1);

but $\quad B : b = AB^2 : ab^2$ (III., 30),

and $\quad H : h = AG : ag = AB : ab$ (VI., 1, Def. 4):

hence $B \times H : b \times h = AB^3 : ab^3 = AG^3 : ag^3$ (III., 1, 12);

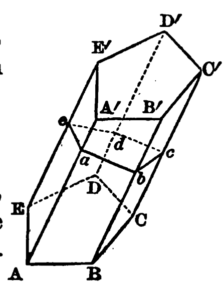
or, ABCDE–G : *abcde–g* $= AB^3 : ab^3 = AG^3 : ag^3$.

COR. Similar prisms are to each other as the cubes of their altitudes.

EXERCISE. The volume of a prism whose edge $AB = 10$ inches is 1,600 cubic inches: what is the volume of a similar prism whose homologous edge $ab = 5$ inches?

IX.

Theorem. *The lateral surface of any prism (cylinder) is equal to the product of the perimeter of a right section by a lateral edge.*



HYPOTH. *abcde* is a right section, and AA′, a lateral edge, of the prism ABCDE–A′.

TO BE PROVED. Lateral surface $= (ab + bc + cd + de + ea)\ AA'$.

PROOF. Since the right section, *abcde*, is perpendicular to the lateral edges, the intersections *ab*, *bc*, &c., are the altitudes of the faces.

Hence $ABB'A' = AA' \times ab,$ (III., 7 Cor.)

$BCC'B' = BB' \times bc = AA' \times bc,$ &c.

Adding these equations, we have,

lateral surface $= (ab + bc + cd + de + ea)\ AA'.$

Cor. 1. The lateral surface of a right prism (cylinder) is equal to the product of the perimeter of its base by its altitude.

Cor. 2. If r equals radius of the base, and A, the altitude, of a cylinder, we have,

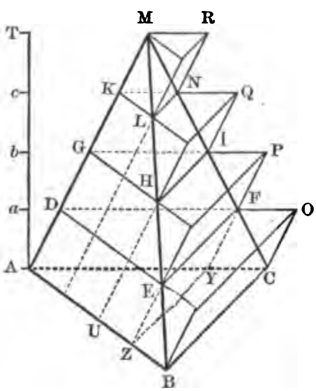
volume of the cylinder $= \pi r^2 A$ (7), (IV., 12 Cor. 3.)

lateral surface of a right cylinder $= 2\pi r A.$

PYRAMIDS.

X.

Theorem. *Upon the base and sections of a triangular pyramid parallel to the base, a series of prisms may be constructed, the sum of whose volumes shall differ from that of the pyramid by less than any assigned volume.*



Hypoth. M–ABC is a triangular pyramid, whose altitude, AT, is divided into the equal parts Aa, ab. . . . DEF, GHI, KLN, are sections of planes passed through the points a, b, c, parallel to the base ABC; and ABC–O, DEF–P, . . . are prisms erected upon the base and the sections, with the equal altitudes, Aa, ab, . . . and the edge MA, taken as an edge of the prisms.

To be Proved. (ABC–O + DEF–P + GHI–Q + KLN–R) — M–ABC is less than the prism ABC–O, which may be made less than any assigned volume.

Proof. Extend the faces, EP, HQ, LR, until they cut the

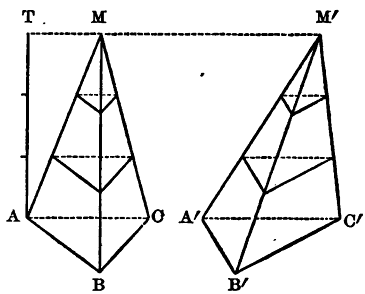
base ABC. They will divide ABC–O into prisms, whose bases are respectively equal to EO, HP, LQ, and MR, and whose altitude is Aa. (VI., 2 Cor. 1.)

Then it is evident that the difference between the prisms, ABC–O and ABC–DEF, the corresponding portion of the pyramid, is less than BCYZ–O, the first division of ABC–O; also DEF–P — DEF–GHI is less than the second division of ABC–O, &c.: hence the sum of all the prisms (ABC–O + DEF–P + GHI–Q + KLN–R) — M–ABC is less than ABC–O, which may be made less than any assigned volume by increasing the number of parts into which the altitude, AT, is divided.

XI.

Theorem. *Two triangular pyramids having equal bases and equal altitudes are equal.*

Hypoth. M–ABC and M′–A′B′C′ are two triangular pyramids having the base ABC = A′B′C′, and the same altitude, AT.



To be Proved. M–ABC = M′–A′B′C′.

Proof. Divide the altitude, AT, into any number of equal parts. Pass planes through the points of division, and construct on the bases and sections prisms as in 10. The prisms of the two pyramids will be equal each to each, since they have the same altitude and equal bases (3, Cor. 1): hence, if P and P′ denote the sums of the two series of prisms, we have

$$P = P'.$$

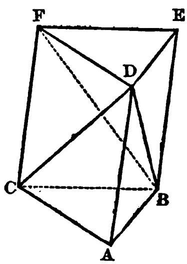
Now, if the number of these prisms be indefinitely increased by increasing the number of equal parts into which the altitudes are divided, the volumes, P and P′, will remain equal,

while they constantly approach the volumes of the pyramids, M–ABC and M′–A′B′C′, from which they will finally differ by less than any assignable volume (10): hence, by the *principle of limits*, M–ABC = M′–A′B′C′.

XII.

Theorem. *Every triangular pyramid is one-third of a triangular prism having the same base and altitude.*

Hypoth. D–ABC is a triangular pyramid. With the base ABC, and edge AD, complete the prism ABC–DEF.



To be Proved. D–ABC $= \frac{1}{3}$ABC–DEF.

Proof. From the prism take away the pyramid D–ABC: there will remain the quadrangular pyramid, whose vertex is D, and base BCFE. Through D and BF, the diagonal of the base, pass a plane dividing D–BCFE into the triangular pyramids D–BCF and D–BEF.

Then D–BCF = D–BEF (11); for they have the same altitude and equal bases (I., 37).

But D–BEF is the same as B–DEF, and B–DEF = D–ABC, having equal bases and the same altitude: hence the whole prism is divided into three equal triangular pyramids, and

D–ABC $= \frac{1}{3}$ABC–DEF.

Cor. The volume of a triangular pyramid is equal to one-third the product of its base by its altitude (7).

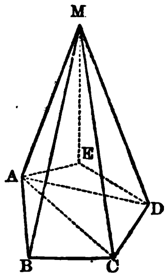
XIII.

Theorem. *The volume of any pyramid is equal to one-third the product of its base by its altitude.*

Hypoth. M–ABCDE is a pyramid whose altitude is A.

To be Proved. M–ABCDE $= \frac{1}{3}$ABCDE $\times$ A.

PROOF. Divide the pyramid into triangular pyramids by passing planes through M, and the diagonals, AC and AD, of the base.



Then M–ABC $=\frac{1}{3}$ABC $\times$ A (XII., Cor.),
M–ACD $=\frac{1}{3}$ACD $\times$ A,
M–ADE $=\frac{1}{3}$ADE $\times$ A.

Adding these equations, member to member, we have M–ABCDE $=\frac{1}{3}$ABCDE $\times$ A.

COR. 1. Pyramids having equal altitudes are to each other as their bases; pyramids having equal bases are to each other as their altitudes; and any two pyramids are to each other as the products of their bases and altitudes.

COR. 2. Similar pyramids are to each other as the cubes of their homologous edges or altitudes (8).

COR. 3. A cone is a regular pyramid, whose base has an infinite number of sides: hence *the volume of a cone is equal to one-third the product of its base by its altitude.*

If R = radius of the base, then volume of the cone $= \frac{1}{3}\pi R^2 \times A$ (IV., 12).

SCHOLIUM. The volume of any polyhedron may be found, if planes be passed through a point within and the edges, dividing it into pyramids. The faces of the polyhedron will be the bases of the pyramids; and the perpendiculars drawn from the point within to the faces will be the altitudes.

XIV.

Theorem. *The lateral surface of a regular pyramid (cone) is equal to one-half the product of the perimeter (circumference) of its base by its slant height (element).*

For the lateral faces are all equal; and each is equal to one-half the product of its base by its altitude (III., 7 Cor.).

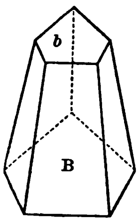
COR. If R = radius of the base, and E = an element of a cone, the surface of the cone $= \pi R \times E$ (IV., 12).

XV.

Theorem. *A frustum of any pyramid is equal to three pyramids, whose bases are the lower base, the upper base, and a mean proportional between the two bases, of the frustum, and whose altitude is the altitude of the frustum.*

Hypoth. B is the lower base, b the upper base, and A the altitude, of a frustum.

To be Proved. Frustum $= (\mathrm{B} + b + \sqrt{\mathrm{B}b})\frac{\mathrm{A}}{3}$ (13).



Proof. Let a denote the altitude of the pyramid which was cut off to form the frustum; then $\mathrm{A} + a$ is the altitude of the completed pyramid; and the

$$\text{entire pyramid} = \frac{\mathrm{B}\,(\mathrm{A}+a)}{3}\ (13);$$

$$\text{pyramid cut off} = \frac{ba}{3}:$$

hence $$\text{frustum} = \frac{\mathrm{B}(\mathrm{A}+a)}{3} - \frac{ba}{3} = \frac{\mathrm{BA}}{3} + \frac{(\mathrm{B}-b)\,a}{3}.$$

But the two pyramids are similar:

hence $\mathrm{B} : b = (\mathrm{A} + a)^2 : a^2$ (VI., 6 Cor. 1);

or, $\sqrt{\mathrm{B}} : \sqrt{b} = \mathrm{A} + a : a:$

hence $a\sqrt{\mathrm{B}} = \mathrm{A}\sqrt{b} + a\sqrt{b};$

or, $a\,(\sqrt{\mathrm{B}} - \sqrt{b}) = \mathrm{A}\sqrt{b}.$

Multiplying both members of this equation by $\sqrt{\mathrm{B}} + \sqrt{b}$, we have,

$$a\,(\mathrm{B}-b) = \mathrm{A}\sqrt{\mathrm{B}b} + \mathrm{A}b.$$

Substituting this value in the above, we have,

$$\text{frustum} = \frac{\mathrm{BA}}{3} + \frac{\mathrm{A}\sqrt{\mathrm{B}b} + \mathrm{A}b}{3} = (\mathrm{B} + b + \sqrt{\mathrm{B}b})\,\frac{\mathrm{A}}{3}.$$

Cor. Let r and r' denote the radii of the bases of a frustum of a cone, and A its altitude: then πr^2 and $\pi r'^2$ are the areas of the bases, and $\pi rr'$ a mean between them (IV., 12 Cor. 3).

$$\text{Hence volume of the frustum} = (r^2 + r'^2 + rr')\,\frac{\pi \mathrm{A}}{3}.$$

EXERCISE 1. If the altitude of a frustum is 6 feet, the bases, 12 and 8 square feet respectively, what is the volume?

EXERCISE 2. If the altitude of a pyramid is 6 feet, and the area of the base 12 square feet, what is the volume of a frustum of that pyramid cut off by a plane 3 feet from the base.

XVI.

Theorem. *The lateral surface of a frustum of a regular pyramid is equal to one-half the sum of the perimeters of its bases multiplied by its slant height.*

For the faces of a frustum of a regular pyramid are equal trapezoids, and each is measured by one-half the sum of its bases by its altitude (III., 11): hence, &c.

COR 1. The lateral surface of a frustum of a right cone is equal to one-half the circumferences of its bases multiplied by its slant height. If r and r' denote the radii of the bases, and s the slant height, then

$$\text{lateral surface} = (r + r')\,\pi s.$$

COR. 2. If a plane be passed through any frustum parallel to and at equal distances from the bases, the perimeter (circumference) of the section multiplied by the slant height is equal to the lateral surface (III., 11, Cor.).

THE SPHERE.

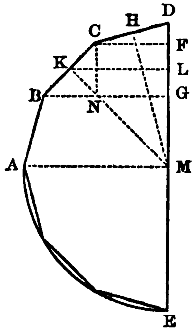
MEASUREMENT OF THE SURFACE.

XVII.

Theorem. *If half a regular polygon of an even number of sides be revolved about a greatest diagonal, the area of the surface generated will be equal to the product of that diagonal and the circumference inscribed within the polygon.*

HYPOTH. ABCD . . . is a regular polygon revolved about the greatest diagonal DE. Let R denote the greater radius, MD, and r the apothgem, MH = MK (IV., 3).

To be Proved. The surface generated by the half-perimeter DAE equals DE $\times$ $2\pi r$ = 2R $\times$ $2\pi r$ = $4\pi Rr$.



Proof. Draw CF and BG $\perp$ DE, CN $\perp$ BG, and from K, the middle point of BC (IV., 2), draw KL $\parallel$ BG.

Then CD will describe the surface of a cone, the radius of whose base is CF, and *area* CD* = π.CF $\times$ CD (14, Cor.).

But $\triangle$ CDF $\sim$ MDH (III., 22);
that is, CF : DF = MH : HD;
or, CF $\times$ 2HD = DF $\times$ 2MH.

Substituting this value, we have,

area CD = π.DF $\times$ 2MH = DF $\times$ $2\pi r$.

Again: BC will describe the surface of a frustum of a cone, the radius of whose middle section is KL: then

area BC = 2π.KL $\times$ BC (16, Cor. 2).

But $\triangle$ KLM $\sim$ CNB (III., 25);
that is, KL : MK = CN : BC;
or, KL $\times$ BC = MK $\times$ CN.

Substituting this value, we have,

area BC = 2π.MK $\times$ CN = FG $\times$ $2\pi r$.

In like manner we may find,

area AB = GM $\times$ $2\pi r$, &c.

Adding these results, we have,

area DAE = DE $\times$ $2\pi r$ = $4\pi Rr$.

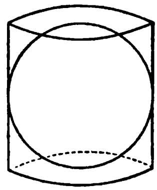
Cor. 1. If the polygon has an infinite number of sides, that is, becomes a circle, the surface generated will be that of a sphere, and r = R (IV., 10, 3): hence *the surface of a sphere* = $4\pi R^2$.

Cor. 2. The area of a great circle = πR^2 (IV., 12, Cor. 3): hence *the surface of a sphere is equal to four times the area of one of its great circles.*

* By *area* CD is meant the area of the surface generated by the line CD when the polygon is revolved.

COR. 3. The surfaces of two spheres are to each other as the squares of their radii.

COR. 4. If a right cylinder be circumscribed about a sphere whose radius is R, then R = radius of its base, 2R = its altitude, and $4\pi R^2$ = its lateral surface (9, Cor. 2); that is, *the surface of a sphere is equal to the convex surface of a circumscribed cylinder.*

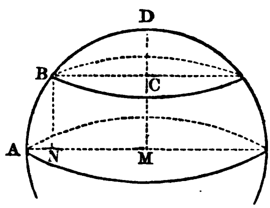


COR. 5. To the lateral surface add the bases of a cylinder, and we have,

$$4\pi R^2 + 2\pi R^2 = 6\pi R^2; \text{ that is,}$$

surface of sphere : *surface of circumscribed cylinder* = 2 : 3.

COR. 6. The surface of the zone generated by the revolution of an arc, AB or BD, is equal to the circumference of a great circle, $2\pi R$, multiplied by its altitude, MC or DC; that is, if A is the altitude of a zone, *the area of the zone* = $2\pi R \times A$. Also, *on the same sphere or equal spheres, two zones are to each other as their altitudes: zones with equal altitudes are equal.*



EXERCISE 1. Two zones on different spheres are to each other as the products of their altitudes and the radii of the spheres.

EXERCISE 2. If the radius of a sphere is 6 inches, what is the area of its surface? What is the area of a zone whose altitude is 3 inches.

EXERCISE 3. Find the surface generated by the revolution of a semi-hexagon about a greatest diagonal when a side is 6 inches.

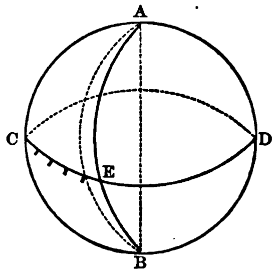
XVIII.

Theorem. *A lune is to the surface of the sphere as the angle of the lune is to four right angles.*

Hypoth. ACBE is a lune; CED a great circle, whose poles are A and B; and S is the surface of the sphere.

To be Proved. The lune ACBE : S = CE : CDE = $\angle$ CAE : 4 right angles.

Proof. Divide the circumference, CED, into any number of equal parts. Through A, B, and the points of division, pass great circles. It is evident that they will divide the sphere into the same number of equal lunes; and the lune CAEB will contain the same number of these lunes as the number of equal parts into which the arc is divided. Hence, whether CE and the circumference CDE are commensurable or incommensurable, it follows (as in II., 9), that the lune



ACBE : S = CE : CDE = $\angle$ CAE : 4 right angles (VII., 11, Cor. 1).

Cor. 1. If L = area of a lune, A = its angle, and S = surface of the sphere; then

$$L : S = A : 4 \text{ right angles};$$

therefore $$L = \frac{S \times A}{4 \text{ right angles}} = \frac{S \times A}{4},$$

when a right angle is the unit of angle;

or, $$L = \frac{4\pi R^2 \times A}{360^\circ} = \frac{\pi R^2 \times A}{90^\circ} \quad (17, \text{Cor. } 1).$$

Cor. 2. On the same sphere, or equal spheres, two lunes are to each other as their angles.

Cor. 3. The same demonstration will give,

Ungula ABCE : sphere = $\angle$ CAE : 4 right angles.

Exercise. What part of the surface of a sphere is a lune whose angle is 60°? 90°? 120°?

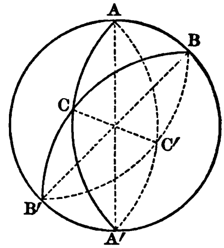
XIX.

Theorem. *The area of a spherical triangle is equal to its spherical excess multiplied by the tri-rectangular triangle; the right angle, or* 90°, *being the unit of angles.*

HYPOTH. ABC is a spherical triangle; $\angle A+B+C-180^\circ$ is its spherical excess; and T a tri-rectangular triangle (VII., 21).

TO BE PROVED. $ABC=\frac{\angle A+B+C-180^\circ}{90^\circ}\times T.$

PROOF. Complete the great circles, ABA′, BCB′, and ACA′. Then from the triangles on the hemisphere, ABA′B′C, we have (by 18, Cor. 1),



$$ABC+A'BC=\frac{\pi R^2\times A}{90^\circ},$$

$$ABC+AB'C=\frac{\pi R^2\times B}{90^\circ},$$

$$ABC+A'B'C=\frac{\pi R^2\times C}{90^\circ}\quad \text{(VII., 15, Cor. 1).}$$

Adding these equations member to member, we have,

$$2ABC+\text{hemisphere}=\frac{\angle A+B+C}{90^\circ}\times\pi R^2.$$

But $\qquad\text{hemisphere}=4T\quad\text{(VII., 21, Cor. 4)},$

and $\qquad\pi R^2=2T\quad\text{(17, Cor. 1)}.$

Substituting these values, transposing, and dividing by 2, we have,

$$ABC=\left(\frac{\angle A+B+C}{90^\circ}-2\right)\times T;$$

or,

$$ABC=\left(\frac{\angle A+B+C-180^\circ}{90^\circ}\right)\times T.$$

EXERCISE 1. What part of the surface of a sphere is the triangle ABC, if $\angle A=120^\circ$, $\angle B=100^\circ$, and $\angle C=95^\circ$? What is its area in square inches, if the radius of the sphere is 6 inches?

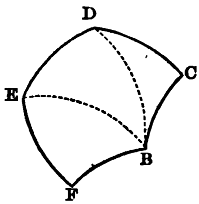
Exercise 2. Show that a spherical triangle cannot be greater than half the surface of the sphere.

XX.

Theorem. *The area of a spherical polygon is equal to its spherical excess multiplied by the tri-rectangular triangle; the right angle, or* 90°, *being the unit of angles.*

Hypoth. BCDEF . . . is a spherical polygon, the sum of whose angles is S, and the number of sides n.

To be Proved. $\text{BCDEF} \ldots = \frac{S-(n-2)180^\circ}{90^\circ} \times T$ (VII., 21).



Proof. Divide the polygon into triangles, by drawing diagonals, BD, BE, . . . from any vertex, B. There will be $n-2$ triangles. Let A′, A″, . . . be the sum of the angles of the triangles, BCD, BDE, . . . respectively.

Then $$\text{BCD} = \frac{A'-180^\circ}{90^\circ} \times T \quad (19),$$

$$\text{BDE} = \frac{A''-180^\circ}{90^\circ} \times T, \text{ \&c.}$$

. . . .

Adding these equations member to member, and remembering that $A' + A'' + \ldots = S$,

we have, $$\text{BCDEF} \ldots = \frac{S-(n-2)\,180^\circ}{90^\circ} \times T.$$

Exercise 1. What is the area of a spherical polygon of five sides, each of whose angles is 120°? 160°?

Exercise 2. The diameter of the earth being 7,912 miles, what is the area of its surface, the earth being considered a perfect sphere? What is the area of a triangle on its surface whose angles are 100°, 120°, and 140°?

Exercise 3. If the surface of a sphere is 100 square inches, what is the surface of a sphere whose radius is twice as great?

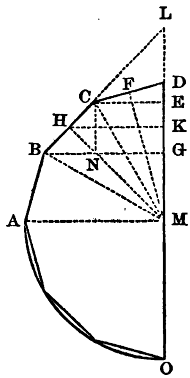
VOLUME OF THE SPHERE.

XXI.

Theorem. *If half a regular polygon of an even number of sides be revolved about a greatest diagonal, the volume generated will be equal to one-third of the surface generated by the sides multiplied by the apothegm of the polygon.*

HYPOTH. DAO is half a regular polygon revolved about the greatest diagonal, DO.

TO BE PROVED. Volume generated $= \dfrac{area \times r}{3} = \dfrac{4\pi R . r^2}{3}$ (17), where $R =$ greater radius, and $r =$ apothegm.



PROOF. Draw the apothegms, MF and MH, the radii MC and MB, and the lines CE, HK, and BG $\perp$ DM; also CN $\perp$ BG. Prolong BC and MD, until they meet in L.

Then the volume generated is equal to the volumes generated by the triangles MCD, MBC, MAB, &c.

But volume MCD is two cones, whose altitudes are DE and EM respectively, and whose common base is generated by CE as a radius: hence

$$vol.\ MCD = \frac{\pi . CE^2 \times DM}{3} = \frac{\pi . CE \times CE \times DM}{3} \quad (13, \text{Cor. } 3).$$

But $$\pi . CE = \frac{area\ CD}{CD} \quad (14 \text{ Cor.}),$$

and CE : MF = CD : DM, since $\triangle$ CDE $\sim$ MDF (III., 22) : hence $$CE \times DM = MF \times CD.$$

Substituting these values, we have,

$$vol.\ MCD = \frac{area\ CD}{CD} \times \frac{MF \times CD}{3} = \frac{area\ CD . r}{3}.$$

It is evident that

$$vol.\ MBC = vol.\ MBL - vol.\ MCL.$$

But $$vol.\ \mathrm{MBL} = \frac{area\ \mathrm{BL}.r}{3},$$

and $$vol.\ \mathrm{MCL} = \frac{area\ \mathrm{CL}.r}{3}:$$

hence $$vol.\ \mathrm{MBC} = \frac{area\ \mathrm{BC}.r}{3}.$$

In like manner we may find,

$$vol.\ \mathrm{MAB} = \frac{area\ \mathrm{AB}.r}{3},\ \&c.$$

Adding these results, we have,

$$vol.\ \mathrm{DAO} = \frac{area\ (\mathrm{DC} + \mathrm{BC} + \mathrm{AB} + \ldots)\ r}{3},$$

and $$vol.\ \mathrm{DAO} = \frac{4\pi\mathrm{R}.r^2}{3}\ (17).$$

COR. 1. If the polygon DAO has an infinite number of sides, that is, becomes a circle, the solid generated will be a sphere, and r will equal R.

Hence $$vol.\ \text{of a sphere} = \frac{4}{3}\pi\mathrm{R}^3;$$

or, if D denotes the diameter,

$$vol.\ \text{of a sphere} = \frac{1}{6}\pi\mathrm{D}^3.$$

COR. 2. *Two spheres are to each other as the cubes of their radii, or diameters.*

COR. 3. *The volume of a spherical sector is equal to the product of the zone which forms its base, by one-third of its radius.*

For if BC is an arc of the circle that generates the sphere, M its centre, and A the altitude CN′, then

$$vol.\ \mathrm{MBC} = \frac{area\ \mathrm{BC} \times \mathrm{R}}{3} = \frac{2\pi\mathrm{RA} \times \mathrm{R}}{3}\ (17,\ \text{Cor. } 6);$$

or, $$vol.\ \mathrm{MBC} = \frac{2}{3}\pi\mathrm{R}^2\mathrm{A}.$$

COR. 4. If R is the radius of the base, and 2R the altitude of a cylinder, we have,

$$vol. \text{ of cylinder} = \pi R^2 \times 2R = 2\pi R^3 \quad (9, \text{Cor. } 2);$$

but *vol.* of a sphere $= \frac{4}{3}\pi R^3$; that is,

vol. of a sphere : vol. of circumscribed cylinder $= 2 : 3$.

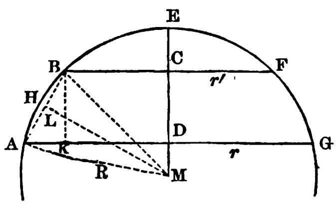
SCHOLIUM. A sphere may be supposed to consist of an infinite number of infinitely small pyramids, whose vertices are at the centre, whose bases form the surface of the sphere, and whose altitude is the radius of the sphere. The volume of these pyramids is equal to one-third the sum of their bases multiplied by their altitude (12, Cor.); or,

$$vol. \text{ of a sphere} = \frac{1}{3} 4\pi R^2 \times R = \frac{4}{3}\pi R^3,$$

which agrees with the result found in Cor. 1.

XXII.

Theorem. *A spherical segment is equal to a cylinder whose base is half the sum of the bases of the segment, and altitude, the altitude of the segment plus a sphere whose diameter is equal to that altitude.*



HYPOTH. r and r' are the radii of the bases, and A the altitude, BK, of a spherical segment, ABFG. R is the radius of the sphere.

TO BE PROVED. $ABFG = \frac{\pi r^2 + \pi r'^2}{2} A + \frac{1}{6}\pi A^3$.

PROOF. The segment is generated by the revolution of AHBCD: then

$$vol. \text{ AHBCD} = vol. \text{ ABCD} + vol. \text{ (AHBM} - \text{ABM)},$$

$$vol. \text{ ABCD} = (2r^2 + 2r'^2 + 2rr')\frac{\pi A}{6} \quad (15, \text{Cor.}),$$

$$vol. \text{ AHBM} = \frac{2}{3}\pi R^2 A \quad (21, \text{Cor. } 3),$$

$$vol. \text{ ABM} = \frac{2LM \times \pi A \times LM}{3} = \frac{2}{3}\pi A \times LM^2 \text{ (21 and 17)}.$$

Substituting these values, we have,

$$vol.\ \text{AHBCD} = (2r^2 + 2r'^2 + 2rr')\,\frac{\pi\text{A}}{6} + \frac{2}{3}\pi\text{A}\,(\text{R}^2 - \text{LM}^2).$$

But since ALM and ABK are right-angled triangles, we have,

$$\text{R}^2 - \text{LM}^2 = \text{AL}^2 = \frac{1}{4}\text{AB}^2 = \frac{1}{4}(\text{BK}^2 + \text{AK}^2) = \frac{1}{4}[\text{A}^2 + (r - r')^2]:$$

henee

$$vol.\ \text{AHBCD} = (2r^2 + 2r'^2 + 2rr')\,\frac{\pi\text{A}}{6} + \frac{\pi\text{A}}{6}[\text{A}^2 + (r - r')^2]$$

$$= (3r^2 + 3r'^2 + \text{A}^2)\,\frac{\pi\text{A}}{6} =$$

$$\left(\frac{\pi r^2 + \pi r'^2}{2}\right)\text{A} + \frac{1}{6}\pi\text{A}^3.$$

Cor. If the segment has but one base, as *vol.* ABED, the radius $r' = \text{O}$, and we have,

$$vol.\ \text{ABED} = \frac{1}{2}\pi r^2\text{A} + \frac{1}{6}\pi\text{A}^3.$$

MODERN GEOMETRY.

BOOK IX.*

ANHARMONIC RATIO.

I.

If A, B, C, and D are four points in a straight line, the ratio

$$\frac{AC}{CB} : \frac{AD}{DB}$$

is called the **anharmonic ratio** of the four points. This ratio is also expressed by [ABCD], which gives the order in which the ratio is written.

The line AB is considered positive, and BA negative. (See p. 2.)

II.

The value of the anharmonic ratio of four points is not changed when two of the points are interchanged, provided the other two are also interchanged.

Thus $[ABCD] = [BADC] = [CDAB] = [DCBA]$;

for $$\frac{AC}{CB} : \frac{AD}{DB} = \frac{BD}{DA} : \frac{BC}{CA} = \frac{CA}{AD} : \frac{CB}{BD} = \frac{DB}{BC} : \frac{DA}{AC}.$$

* This book is designed for those students who have time and desire to learn the elements of modern geometry in addition to what has, until recently, been given in our text-books of elementary geometry. The following authors are recommended to those who wish to pursue the subject further, — Mulchay, Townsend, Salmon, Rouché and Comberousse, Chasles, Poncelet, Poinsot, and Steiner.

Any other changes in the order of points will change the value of the ratio. The four letters may be written in twenty-four different orders; and hence, from four points in a line, six different anharmonic ratios may be formed.

(Write the six anharmonic ratios, and show that three are the reciprocals of the other three.)

III.

The anharmonic ratio $\frac{AC}{CB} : \frac{AD}{DB}$ is *positive* when the two points C and D are both between A and B, or both without; and *negative*, when one is between, and the other without.

If ABCD is the order of the points in the line, and AB is supposed to be cut by C and D, or AD is supposed to be cut by B and C, the anharmonic ratios are positive.

If AC is supposed to be cut by B and D, the anharmonic ratio is negative. The reciprocals of these have the same signs. Hence, *of the six anharmonic ratios of four points, four are positive, and two are negative.*

$$\frac{AC}{CB} : \frac{AD}{DB} = \infty \text{ when } CB = 0, \text{ or } AD = 0.$$

$$\frac{AC}{CB} : \frac{AD}{DB} = 0 \text{ when } AC = 0, \text{ or } DB = 0.$$

$$\frac{AC}{CB} : \frac{AD}{DB} = +1 \text{ when } AB = 0, \text{ or } CD = 0.$$

That is, *if two points coincide with each other, the anharmonic ratio takes one of the values*, ∞, 0, $+1$.

$$\frac{AC}{CB} : \frac{AD}{DB} = -1. \quad \text{(See Harmonic Proportion.)}$$

IV.

If one of the points, as C, bisects AB, or is at an infinite distance, AC = CB, and the anharmonic ratio

$$\frac{AC}{CB} : \frac{AD}{DB} = \frac{DB}{AD};$$

also the five remaining anharmonic ratios reduce to simple ratios.

EXERCISE. If P, Q, R, $\frac{1}{P}$, $\frac{1}{Q}$, and $\frac{1}{R}$ are the six anharmonic ratios of four points, show that

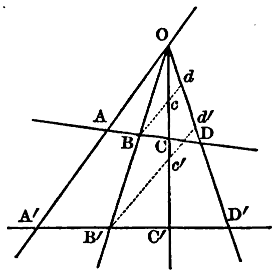
$$P \times Q \times R = -1,$$

and

$$\frac{1}{P} \times \frac{1}{Q} \times \frac{1}{R} = -1.$$

V.

DEF. 1. A system of lines passing through a common point is called a **pencil**. Each line is called a **ray**; and the common point, the **vertex**.



Theorem. *If a pencil of four rays is cut by any transversal, the anharmonic ratio of the points of intersection is constant for all positions of the transversal.*

HYPOTH. O–ABCD is a pencil cut by the transversals ABCD and A′B′C′D′.

TO BE PROVED. [ABCD] = [A′B′C′D′].

PROOF. Draw Bd and B′d' ∥ OA.

Then △ OAC ∼ cBC, and △ OAD ∼ dBD (III., 22):

hence $$\frac{AC}{CB} = \frac{OA}{cB}, \text{ and } \frac{AD}{DB} = \frac{OA}{dB}.$$

Dividing these equations member by member, we have,

$$[ABCD] = \frac{dB}{cB}.$$

In like manner we may find,

$$[A'B'C'D'] = \frac{d'B'}{c'B'};$$

but $$\frac{dB}{cB} = \frac{d'B'}{c'B'}:$$

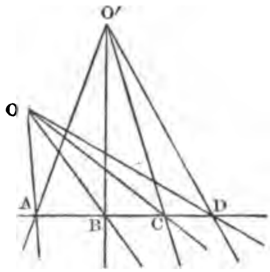
hence $$[ABCD] = [A'B'C'D'].$$

SCHOLIUM. The anharmonic ratio is the same when the transversal cuts one or more of the rays on the other side of

the vertex. (Draw the figure, and apply the same demonstration.)

Def. 2. The anharmonic ratio [ABCD] of the four points in the transversal is also called the **anharmonic ratio of the pencil**, which may be expressed by the form [O–ABCD].

Cor. 1. *If two pencils are mutually equiangular, they have the same anharmonic ratio.*



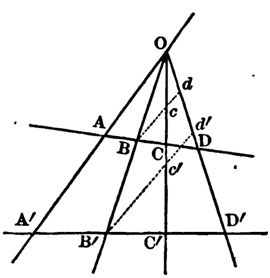
Cor. 2. *If the intersections of the rays of two pencils are in the same straight line, they have the same anharmonic ratio.*

Cor. 3. *If two pencils have the same anharmonic ratio, it does not follow that they are mutually equiangular.*

VI.

Problem. *Given three rays, the anharmonic ratio, and the relative position of the fourth ray, of a pencil, to find the exact position of the fourth ray.*

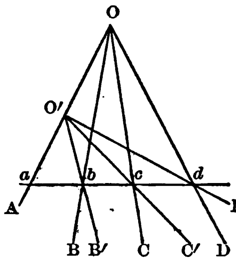
Given OA, OB, OC, and [ABCD]; also OD lies to the right of OC. Find the exact position of OD.



Through B draw Bcd parallel to OA. Then $[ABCD] = \frac{dB}{cB}$ (5), which fixes the position of d and of the ray OdD.

Cor. D is a fixed point in the transversal ABC when the anharmonic ratio [ABCD] is given. Hence, also, the ray OD is fixed. From this follow the two corresponding theorems, VII. and VIII.

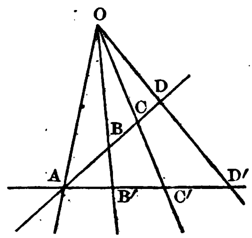
VII.



Theorem. *If two pencils*, O–ABCD *and* O′–A′B′C′D′, *have a homologous ray*, OA, *common, and the same anharmonic ratio, the intersection*, d, *of a pair of homologous rays is in the same straight line with* b *and* c, *the intersections of the remaining two pairs of rays.*

VIII.

Theorem. *If, in two straight lines, four points*, A, B, C, D, *in the one, have the same anharmonic ratio as four points*, A, B′, C′, D′, *in the other, and a homologous point*, A, *in common, the straight lines passing through the other pairs of homologous points meet in a common point*, O.



For draw B′B, C′C, and produce them until they meet in O; join OA; then the fourth ray is fixed, and it determines the points D and D′ (6).

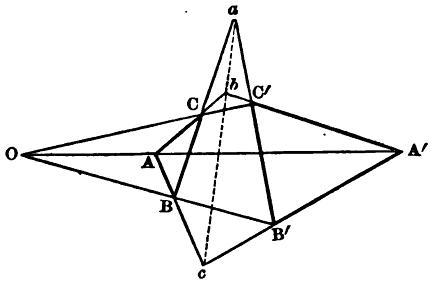
IX.

Theorem. *If two triangles*, ABC *and* A′B′C′, *in space be so situated that the lines*, AA′, BB′, *and* CC′, *joining corresponding vertices meet in a common point*, O, *the intersections* a, b, c, *of the corresponding sides are in the same straight lines;* and conversely.

Proof. The lines CC′ and BB′ are in the same plane, since they meet in a point O (V., 1): hence BC and B′C′ are in the same plane, and will meet in *a*, a point of the

intersection of the planes of the triangles ABC and A′B′C′. (V., 27.)

In like manner it may be shown that *b* and *c* are in the same line of intersection.



Conversely the points *a*, *b*, and *c*, are in the same straight line.

TO BE PROVED. AA′, BB′, and CC′ will meet in a common point, O.

PROOF. BC and B′C′ are in the same plane, since they meet in a point, *a*: hence CC′ and BB′ are in the same plane, and will meet in a point, O, of the intersection of the planes of the triangles *b*CC′ and *c*BB′. In like manner it may be shown that A and A′ are in the same line.

COR. This theorem is true, whatever angle the planes of the triangles form with each other. It is therefore true when this angle is reduced to 0°, or 180°, by turning the planes about the line of intersection: hence it is true when the triangles are in the same plane.

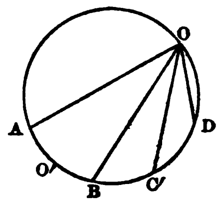
ANHARMONIC PROPERTIES OF THE CIRCLE.

X.

DEF. The anharmonic ratio of any four points, A, B, C, D, on the circumference of a circle, is the anharmonic ratio of the pencil formed by lines joining each of those points with a fifth point, O, on the circumference.

Theorem. *The anharmonic ratio of four fixed points on the circumference of a circle is constant.*

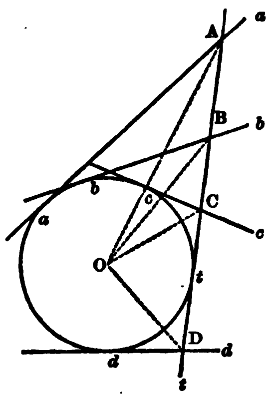
For, all the pencils drawn from points in the arc AOD are mutually equiangular (II., 11, Cor. 2). If the vertex is at O′ in the arc AB, the pencil is mutually equiangular with one whose rays are OB, OC, OD, and the prolongation of OA through O: hence the anharmonic ratio remains the same (5, Schol.).



XI.

DEF. The *anharmonic ratio of four tangents* to a circle is the anharmonic ratio of the points of intersection of those tangents with a variable fifth tangent.

Theorem. *The anharmonic ratio of four tangents to a circle is constant.*



HYPOTH. a, b, c, and d are four fixed tangents cut by the variable tangent t in the points A, B, C, and D.

TO BE PROVED. [ABCD] has the same value for all positions of t.

PROOF. From the centre, O, draw OA, OB, OC, and OD; and suppose radii drawn to the points of contact a, b, c, d, and t; then, for any position of t,

$$\angle \text{AOB} = \tfrac{1}{2}(a\text{O}t - b\text{O}t) = \tfrac{1}{2}a\text{O}b. \quad \text{(Show this.)}$$

$$\angle \text{BOC} = \tfrac{1}{2}(b\text{O}t - c\text{O}t) = \tfrac{1}{2}b\text{O}c,$$

$$\angle \text{COD} = \tfrac{1}{2}(c\text{O}t + d\text{O}t) = \tfrac{1}{2}c\text{O}d.$$

That is, the angles of the pencil O–ABCD have the same values for all positions of t: hence [ABCD] is constant.

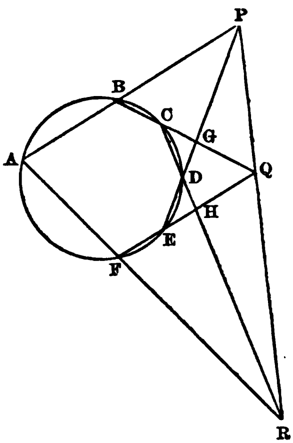
COR. The anharmonic ratio of four tangents, a, b, c, and d, is equal to the anharmonic ratio of their points of contact, a, b, c, and d. (Show this.)

XII.

PASCAL'S THEOREM.

Theorem. *If a hexagon is inscribed in a circle, the intersections of the three pairs of opposite sides are in the same straight line.*

Hypoth. ABCDEF is an inscribed hexagon. P, Q, and R are the intersections of the opposite sides.

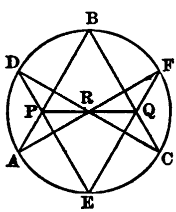


To be Proved. PQR is a straight line.

Proof. Suppose the lines BD, BE, FD, and FC drawn. Then [B–AEDC] = [F–AEDC] (10); and, since PE and RC cut these pencils, we have [PEDG] = [RHDC].

But PE and RC have the homologous point D common: hence the lines PR, EH, and GC, meet in one point, Q (8); and PQR is a straight line.

Scholium. The same demonstration proves the theorem true for six points in the circumference, taken in any order.



Cor. 1. If the side BC gradually diminish, until B and C coincide, QB will be a tangent, and the inscribed figure a pentagon: hence, *if a pentagon be inscribed in a circle, the intersection of a tangent at a vertex with the opposite side is in the same straight line with the two intersections of the remaining non-consecutive sides.* (Construct the figure.)

Cor. 2. If, in like manner, the opposite side QF also become a tangent, the inscribed figure will be a quadrilateral: hence, *if a quadrilateral be inscribed in a circle, the intersec-*

tions of the two pairs of tangents at opposite vertices are in the same straight line with the intersections of the two pairs of opposite sides. (Construct the figure.)

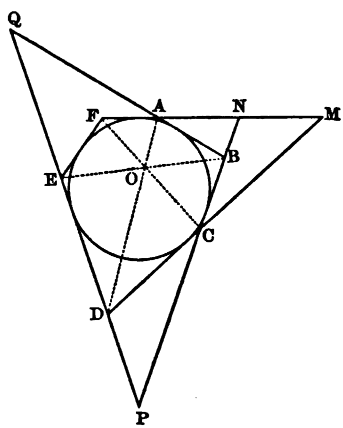
COR. 3. If QB and PE, two alternate sides, become tangents, we have the theorem, *if a quadrilateral be inscribed in a circle, and tangents drawn to two consecutive vertices, the intersection of each of them with the side passing through the point of contact of the other, and the intersection of the other two sides, are three points in the same straight line.* (Construct the figure.)

COR. 4. If three alternate sides produced become tangents, the hexagon becomes a triangle: hence, *if a triangle be inscribed in a circle, the intersections of the tangents at the vertices with the opposite sides are in the same straight line.* (Construct the figure.)

XIII.

BRIANCHON'S THEOREM.

Theorem. *The diagonals joining the opposite vertices of a circumscribed hexagon intersect in the same point.*



HYPOTH. ABCDEF is a circumscribed hexagon.

TO BE PROVED. The diagonals AD, BE, and CF, intersect in the same point, O.

PROOF. Suppose AB, BC, CD, and EF, four fixed tangents cut by ED in the points Q, P, D, E, and by FA in the points A, N, M, F: then

[QPDE] = [ANMF] (11):

hence

[B–QPDE] = [C–ANMF].

Also these pencils have a common ray, NP. Therefore the intersections, A, O, D, of

the three remaining pairs of rays are in the same straight line, AD (7); that is, the diagonals AD, BE, and CF, intersect in the same point, O.

Scholium. Four corollaries follow from this theorem, similar to those in the preceding article. The number of sides is diminished by increasing an angle A, until the sides BA and AF are in the same straight line, and A is a point of contact. (Give the corollaries, and construct the figures.)

HARMONIC PROPORTION.

XIV.

Def. 1. When an anharmonic ratio, $\frac{AC}{CB} : \frac{AD}{DB}$, has the particular value, -1, the points A, B, C, and D, are called **harmonic points**, and the line AB is said to be divided **harmonically** in the points C and D.

$$\frac{AC}{CB} : \frac{AD}{DB} = -1, \text{ or, } \frac{AC}{CB} = -\frac{AD}{DB},$$

agrees with Definition, p. 82, when the direction of the lines is considered.

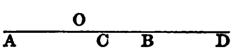
Def. 2. A **harmonic pencil** is a pencil of four rays, whose anharmonic ratio is equal to -1.

Cor. *Any transversal cutting a harmonic pencil is divided harmonically* (5).

XV.

Theorem. *If a straight line*, AB, *is divided harmonically in the points* C *and* D, *and* O *is the centre of* AB, *then*

$$OB^2 = OC \times OD;$$



for, since AC : CB = AD : DB,

we have, by composition and division,

$$AC - CB : AC + CB = AD - DB : AD + DB;$$

or, $$2OC : 2OB = 2OB : 2OD:$$

hence $$OB^2 = OC \times OD.$$

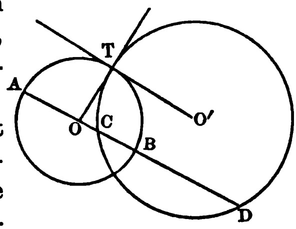
Cor. Conversely, *if* O *is the centre of* AB, *and* $OB^2 =$

OC $\times$ OD, *the line* AB *is divided harmonically in* C *and* D. (Show this.)

XVI.

Theorem. *If two circles cut each other* **orthogonally** (*that is, so that the tangents at a point of intersection are perpendicular to each other*), *any line passing through the centre of one circle, and cutting the other, is divided harmonically.*

The tangent, OT, to one circle, passes through the centre, O, of the other (II., 11, Cor. 8); then $OT^2 = OB^2 = OC \times OD$ (III., 37, Cor. 2): hence AB is divided harmonically in C and D (15 Cor.).

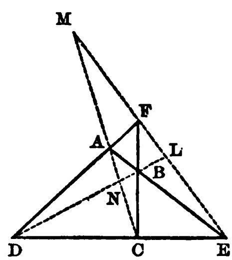


COR. Conversely, if AB is cut harmonically in C and D, any circle passing through the conjugate points, C and D, is cut orthogonally by the circle whose diameter is AB.

XVII.

Theorem. *In a complete quadrilateral, each diagonal is divided harmonically by the other two* (I., 36).

In the complete quadrilateral, ABCDEF, the diagonals, AC, DB, and EF, cut each other harmonically in the points L, M, N.



PROOF. In the triangle FDE, the lines FC, DL, and EA, pass through a common point, B: hence

$AF \times DC \times EL = AD \times CE \times LF$ (III., 35).

Also, since the transversal CAM cuts the triangle DEF, we have,

$$AD \times CE \times MF = AF \times DC \times EM \text{ (III., 33).}$$

Multiplying these equations, member by member, we have,

$$EL \times MF = EM \times LF:$$

hence $$EL : LF = EM : MF.$$

The lines DE, DL, DF, and DM, evidently form a harmonic pencil (14), and the line MC is cut harmonically (14, Cor.); that is,

$$CN : NA = CM : MA.$$

Also FM, FA, FN, and FC form a harmonic pencil: hence the line DL, which cuts them (produced), is divided harmonically; that is,

$$DN : NB = DL : LB.$$

POLES AND POLARS WITH RESPECT TO A CIRCLE.

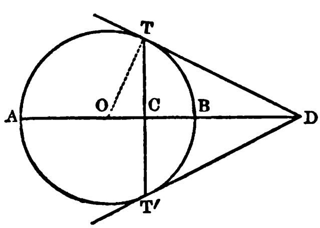
XVIII.

Def. The perpendicular, P, drawn through one of two conjugate harmonic points, C and D, on the diameter of a circle, is called the **polar** of the other point; conversely, this point is the **pole** of the line P (Fig. p. 199).

From III., 19, Cor. 2, it follows, 1st, If the pole is within the circle, the polar is without, and conversely; 2d, As the one approaches the centre, the other recedes to an infinite distance from the circle; 3d, The pole and polar approach the circumference at the same time, and the pole of a tangent is its point of contact to the circle.

XIX.

Theorem. *If from a point,* D, *without a circle, two tangents be drawn, the line* TT′, *passing through the points of contact, is the polar of the given point.*



For TT′ ⊥ AD (II., 18, Cor. and 14); also △ OTC ∼ OTD (III., 22 and II., 11, Cor. 6):

hence $OD : OT = OT : OC$;

or, $OT^2 = OB^2 = OC \times OD$,

and AB is divided harmonically, in the points C and D (15, Cor.): hence TT′ is the polar of the point D.

This theorem furnishes an easy solution to the following problems.

XX.

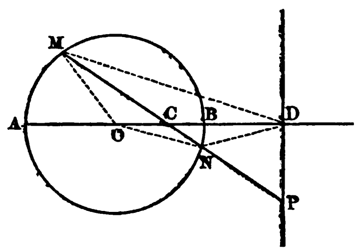
PROBLEM. *Given a point to construct a polar with respect to a circle, and, conversely, given a straight line to find its pole with respect to a circle.* (Give solutions of the four cases.)

XXI.

PROBLEM. *Given three points in a straight line to find the fourth harmonic point.*

XXII.

Theorem. *If, about a point in the plane of a circle, a secant be turned, the polar of the fixed point will be the locus of the fourth harmonic point.*



HYPOTH. DP is the polar of the fixed point C, and MNP is any position of the secant turned about C.

TO BE PROVED. P is the fourth harmonic point of M, C, and N.

PROOF. Draw ON, OM, DN, and DM.

Then $ON^2 = OC \times OD$ (15):

hence $OC : ON = ON : OD$.

and $\triangle OCN \sim OND$ (III., 23):

hence $\angle ONC = ODN$.

In the same manner it may be shown that

$\angle OMC = ODM$;

but $\angle ONC = OMC$ (I., 25):

hence $ODN = ODM$;

and the complements of these angles are equal. Therefore DA and DP bisect the angles made by MD and ND, and hence M, C, N, and P are four harmonic points (III., 19).

If the fixed point C is without the circle, and its polar within, the demonstration is the same, except that the angles ONC and OMC are supplements of each other.

Cor. 1. *If any number of points on a straight line be taken as poles, their polars with respect to a circle pass through a common point, which is the pole of the given line, and conversely.* For any point P on a line DP is the conjugate harmonic point of C, the pole of DP.

Cor. 2. *If, from any number of points on a straight line, pairs of tangents to a circle be drawn, the chords joining their point of contact pass through one common point, which is the pole of the given straight line, and conversely* (19).

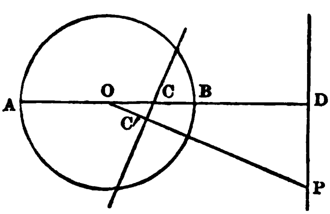
Hence, *as the pole moves on a straight line*, DP, *its polar, revolves about a fixed point*, C, *and conversely.*

Exercise. If the pole moves on a line passing through the centre of the circle, where is the fixed point about which the polar revolves? If DP is a tangent, where are the points M, N, and C? If DP passes through the centre of the circle, construct those points.

The following is another demonstration of Cor. 1.

XXIII.

Theorem. *The polars of all the points of a straight line pass through a common point, which is a pole of that line.*



Let DP be a line, P any point in the line, and C its pole with respect to the circle whose centre is O. Draw PO, and $CC' \perp PO$.

Then $\triangle OC'C \sim ODP$: hence

$OC' \times OP = OC \times OD = OB^2$,

and CC′ is the polar of the point P (15, Cor. and 18). In like manner it may be shown that the polar of any other point in DP passes through the pole C.

XXIV.

Theorem. Conversely, *if any number of lines pass through one point, the locus of their poles is the polar of the point.*

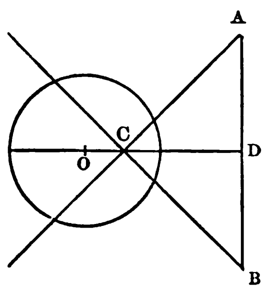
Let CC′ be any line passing through the point C, whose polar is DP. Draw OC′P ⊥ CC′. Then (as in 23) we find OC′ × OP = the square of the radius: hence P is the pole of CC′; and the pole of any line passing through C is in its polar, PD.

XXV.

Since two points determine the position of a straight line, it follows from the preceding articles, that *the line joining any two points is the polar of the intersection of the polars of those points*, and, conversely, *the intersection of two lines is the pole of the line joining the poles of those lines.*

COR. 1. *If two sides of a triangle are the polars of the opposite vertices with respect to a circle, the third side is the polar of the opposite vertex with respect to the same circle.*

For if A is the pole of BC, and B is the pole of AC, the intersection, C, is the pole of AB.



DEF. Any triangle, ABC, whose vertices are the poles of the opposite sides, is said to be **self-reciprocal** with respect to the circle.

COR. 2. It is evident from the definition of poles and polars, that *the altitudes of a self-reciprocal triangle pass through the centre of the circle.*

RECIPROCAL POLARS.

XXVI.

Theorem. *If two polygons are so related to each other with respect to a circle, that every vertex of one is the pole of a corresponding side of the other, or conversely, then each vertex of the latter is the pole of a corresponding side of the former.*

For if A, B, C, D, &c., the vertices of one polygon, are the poles of a, b, c, d, &c., of the other, respectively, then AB is the polar of the intersection of a and b, &c. (25).

Def. Two polygons thus related to each other with respect to a circle are called **reciprocal polars**; and the circle is called the **auxiliary circle.**

Cor. 1. Since the pole of a tangent is its point of contact, it follows that *the reciprocal polar of an inscribed polygon is the circumscribed polygon formed by drawing tangents at the vertices.*

Cor. 2. It is evident, from the definition of poles and polars, that *the lines drawn from the vertices*, A, B, C, D, &c., *of either one of two reciprocal polars perpendicular to* a, b, c, d, &c., *the sides of the other, will pass through the centre of the auxiliary circle.* (Construct the figure.)

Cor. 3. The theorem is true when the polygons have an infinite number of sides; that is, when they are curves. The infinitesimal sides prolonged are tangents: hence, *if any two curves are so related to each other with respect to a circle, that every point of one is the pole of a corresponding tangent to the other, and conversely, then every point of the latter is the pole of a corresponding tangent of the former.*

Exercise 1. Two concentric circles are reciprocal polars with respect to a third concentric circle, when the radius of the latter is a geometrical mean between the radii of the former.

Exercise 2. If A, B, C, D, &c., are the vertices of a polygon, and a, b, c, d, &c., the corresponding sides of its recip-

rocal polygon with respect to a circle, find the pole of the diagonal AD.

XXVII.

The relation between two reciprocal polars is such that every line of the one has a corresponding point in the other, and conversely: hence it follows, that, *from any theorem in relation to the* $\left[\genfrac{}{}{0pt}{}{\textit{lines}}{\textit{points}}\right]$ *of one figure, there follows reciprocally a corresponding theorem in relation to the* $\left[\genfrac{}{}{0pt}{}{\textit{points}}{\textit{lines}}\right]$ *of the other.*

By constructing an inscribed hexagon and its reciprocal polar (26, Cor. 1), it will be seen that Brianchon's theorem (13) and corollaries follow reciprocally from Pascal's theorem and corollaries (12), and conversely. For the points P, Q, and R (see fig. in 12) are the poles of the three diagonals joining the opposite vertices of the reciprocal polar of ABCDEF. Hence these diagonals all pass through the pole of PQR (23).

XXVIII.

Any two reciprocal polars possess the following properties: —

1. Every line of one is perpendicular to the line joining the corresponding point of the other, and the centre of the circle; and conversely.

2. The angle formed by two lines is equal to (supplement of) the angle contained by the two lines drawn from their poles to the centre of the auxiliary circle (I., 35, Cor. 5); and conversely.

3. The product of the distances of any line of one and the corresponding point of the other from the centre of the auxiliary circle is constant; and conversely (15).

4. If two points of one are equidistant from the centre of the auxiliary circle, the corresponding lines of the other are also equidistant from the centre of the circle.

5. If three or more points of one are in a straight line, the

corresponding lines of the other pass through a common point.

From 5 and 2 there follows, —

6. If four points are in a straight line, their polars with respect to a circle form a pencil which has the same anharmonic ratio as the points.

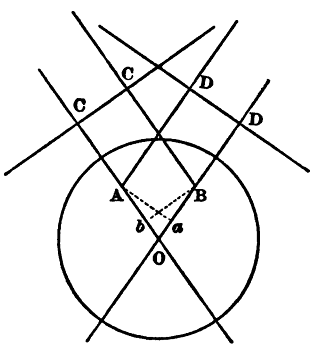
XXIX.

When a circle is concentric with the auxiliary circle, its reciprocal polar is also a circle. When it is not concentric with the auxiliary circle, the reciprocal polar is a figure of a more general form: hence *we may infer from simple properties of the circle all the reciprocal properties of the more complicated figures into which the circle may be transformed by the process of* **reciprocation.**

XXX.

SALMON'S THEOREM.

Theorem. *The distances of any two points from the centre of the auxiliary circle have the same ratio as their distances each from the polar of the other.*



HYPOTH. A and B are two points; C and D their polars with respect to the circle whose centre is O; AD and BC their distances from the polars.

TO BE PROVED. AO : BO = AD : BC.

PROOF. Draw $Bb \perp OC$, and $Aa \perp OD$.

Then $OA \times OC = OB \times OD =$ square of radius,
whence $AO : BO = OD : OC = Oa + AD : Ob + BC$;
but $AO : BO = Oa : Ob$ (III., 22):
hence $AO : BO = AD : BC$.

Construct the figures when one or both the points A and B are without the auxiliary circle.

RADICAL AXIS OF CIRCLES.

XXXI.

The power of a point with respect to a circle = product of the segments of the secants drawn through the point = the square of the distance of the point from the centre minus the square of the radius = the square of the tangent drawn from the point to the circle (III., 37, and corollaries).

Def. If P be a point in the line AB joining the centres of two circles, whose radii are R and r, such that $AP^2-BP^2 = R^2-r^2$, the line PD $\perp$ AB is called the **radical axis** of the circles. (See next fig.)

Cor. The powers of the point P with respect to the two circles are equal. For $AP^2 - R^2 = BP^2 - r^2$.

From the definition of the radical axis, it is evident that, —

1. If the circles are equal, and not concentric, it bisects the line joining the centres.

2. If the circles are concentric, and not equal, it is at an infinite distance.

3. If the circles are concentric and equal, it is indeterminate.

4. If the circles intersect, it passes through the two points of intersection (II., 14).

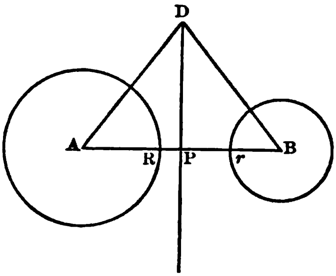
5. If the circles touch each other, it is their common tangent.

XXXII.

Theorem. *The locus of all the points whose powers with respect to two circles are equal is the radical axis of the circles.*

Hypoth. R and r are the radii of two circles, whose centres are A and B. D is any point whose distances DA and DB are such that $DA^2 - R^2 = DB^2 - r^2$.

To be Proved. D is a point of the radical axis.



Proof. Draw DP $\perp$ AB.

Then $DA^2 - R^2 = DB^2 - r^2$ (Hypoth.): whence

$$DA^2 - DB^2 = R^2 - r^2.$$

Also in the right-angled triangles DAP and DBP, we have,

$$AP^2 - BP^2 = DA^2 - DB^2$$
$$= R^2 - r^2$$

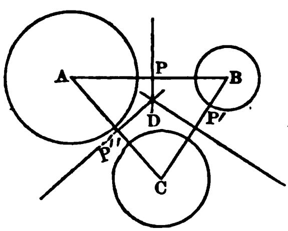
Hence PD is the radical axis of the two circles, and the variable point D is its locus.

XXXIII.

Theorem. *The radical axes of three circles, taken two and two, meet in one point.*

Hypoth. A, B, and C are three circles, which may occupy any relative positions.

To be Proved. The radical axes of A and B, of B and C, and of A and C, meet in one point D.



Proof. Draw the radical axes DP and DP′, of the first two pairs of circles. They will meet in a point D. The powers of the point D, with respect to the circles A and B, are equal; also those with respect to B and C are equal. Hence the powers of the point D, with respect to A and C, are equal, and D is a point of the radical axis of A and C (32).

Def. The point D, at which the radical axes of three circles meet, is called their **radical centre.**

Cor. 1. *If the centres of the circles are in the same straight line, the radical centre is at an infinite distance.*

Cor. 2. *The six tangents, real or imaginary, drawn to three circles from their radical centre, are equal; and conversely.*

Cor. 3. *If from* D, *as a centre, with a radius* DT, *one of the tangents to the circle* A, *a circumference be drawn, it will cut the three circles orthogonally* (16).

For $DT^2 = DA^2 - AT^2$; whence $\angle$ ATD = R (III., 14, Cor.); and the same is true of the other circles.

Cor. 4. If from the radical centre D, secants be drawn cutting the three circles, the products of their segments are equal (30).

Cor. 5. *The radical axis* DP, *of any two circles,* A *and* B, *is the locus of the centres of all the circles that cut* A *and* B *orthogonally;* and *the line* AB, *joining the centres of two circles, is the radical axis of any two and all the circles which cut* A *and* B *orthogonally,* that is, *whose centres lie in* DP.

For the tangent from the centre of any one of those circles to A or B is the radius of that circle; and the tangent from the centre A or B to one of those circles is the radius of A or B: hence, &c. (Construct the figure.)

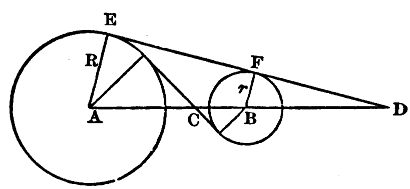
Cor. 6. *If any two circles which cut two other circles,* A *and* B, *orthogonally, intersect each other, their points of intersection are two fixed points in the line* AB, *joining the centres of the latter.*

For the radical axis AB, of the former (Cor. 5) is their common chord (31, 4).

CENTRES OF SIMILITUDE.

XXXIV.

Def. If the line AB, joining the centres of two circles, is divided harmonically in the ratio of the two radii, the two conjugate points, C and D, are called respectively **the internal** and **the external centres of similitude.** Thus if AC : CB = AD : DB = R : r,



C and D are centres of similitude of the circles A and B.

From the definition of centres of similitude, it is evident that, —

1. If the circles are equal, and not concentric, the one bisects the line, the other is at an infinite distance.

2. If the circles are concentric, and not equal, they both coincide with the common centre.

3. If the circles are both concentric and equal, the one is at the common centre, the other is indeterminate.

4. If the circles intersect each other, the lines joining them with a point of intersection bisect the angles formed by the radii drawn to the same point (III., 19). Construct figure.

5. If the circles touch each other, the point of contact is one centre of similitude.

XXXV.

Theorem. *If, in two circles, two radii are drawn, parallel to each other, the line joining their extremities passes through the external centre of similitude if the radii are in the same direction, and the internal centre of similitude if the radii are in opposite directions.*

For, if AE || BF, the triangle DAE $\sim$ DBF;

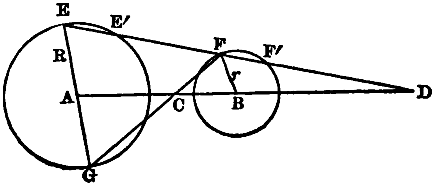
whence $AD : DB = R : r$.

Also, if AG || BF, the triangle AGC $\sim$ BFC;

whence $AC : CB = R : r$.

Hence C and D are the centres of similitude.

Def. The points E and F, also E′ and F′, are called **homologous** points: E and F′, also E′ and F, are called **anti-homologous** points.



Cor. 1. Conversely, *if any transversal is drawn from a cen-*

tre of similitude of two circles, the radii drawn to two homologous points are parallel. (Show this.)

COR. 2. If the transversal DE be turned about the point D until E and E′ coincide, F and F′ will coincide at the same time (fig. in 34).

Hence, *if a line be drawn from either centre of similitude, tangent to one of the circles, it will be tangent to the other also.*

DE : DF and DE′ : DF′ are constant positive ratios, being equal to R : *r*; the rectangle DE $\times$ DF′ $=$ DE′ $\times$ DF for all positions of the transversal turned about D; and the same is true of the transversal turned about C, except that the ratios and rectangles are negative. Hence it follows, that, —

If on a line revolving about a fixed point D (or C), *and intersecting a fixed circle* B, *two variable points*, E *and* E′, *be taken, such that* DE $\times$ DF′ $=$ DE′ $\times$ DF $=$ *a constant quantity, the locus of the two points* E *and* E′ *is another circle*, A, *of which and the circle* B, *the point* D *is a centre of similitude; external if the product* DE $\times$ DF′ *is positive, and internal if that product is negative.*

TEXT-BOOKS ON GOVERNMENT.

ALDEN'S CITIZEN'S MANUAL.

185 Pages.

A Text-Book on Civil Government, in connection with American Institutions.

By JOSEPH ALDEN, D. D., LL. D., President of the State Normal School, Albany, N. Y.

This book was prepared for the purpose of presenting the subjects of which it treats in a manner adapted to their study in Common Schools. It has been extensively adopted, and is widely used, with most gratifying results. It is introductory to this author's larger book.

THE SCIENCE OF GOVERNMENT,

In connection with American Institutions. 295 pages.

By Dr. ALDEN. Intended as a text-book on the Constitution of the United States for High Schools and Colleges. This book contains in a compact form the facts and principles which every American citizen ought to know. It may be made the basis of a brief or of an extended course of Instruction, as circumstances may require.

SPELLERS.

PATTERSON'S COMMON SCHOOL SPELLER.

160 Pages.

By CALVIN PATTERSON, Principal Grammar School No. 13, Brooklyn, N. Y.

This book is divided into seven parts, and thoroughly graded.

PATTERSON'S SPELLER AND ANALYZER.

176 Pages.

Designed for the use of higher classes in schools and academies.

This Speller contains a carefully selected list of over 6,000 words, which embrace all such as a graduate of an advanced class should know how to spell. Words seldom if ever used have been carefully excluded. The book teaches as much of the derivation and formation of words as can be learned in the time allotted to Spelling.

PATTERSON'S BLANK EXERCISE BOOK.

For Written Spelling. Small size. Bound in stiff paper covers.

40 Pages.

PATTERSON'S BLANK EXERCISE BOOK.

For Written Spelling. Large size. Bound in board covers.

72 Pages.

Each of these Exercise Books is *ruled, numbered,* and *otherwise arranged* to *correspond with the Spellers. Each book contains directions by which written exercises in Spelling may be reduced to a system.*

There is also an *Appendix*, for *Corrected Words*, which is in a *convenient form* for *reviews.*

By the use of these Blank Exercise Books a class of four hundred may, in thirty minutes, spell fifty words each, making a total of 20,000 words, and carefully criticise and correct the lesson; each student thereby receiving the benefit of spelling the entire lesson and correcting mistakes.

LOSSING'S
HISTORIES OF THE UNITED STATES.

Lossing's Primary History of the United States.
238 pages..
For the youngest scholars, and illustrated with numerous engravings. By BENSON J. LOSSING, LL.D.

Lossing's Outline History of the United States.
400 pages..

In ***elegance of appearance*** and ***copious illustrations,*** both by pictures and maps, the OUTLINE HISTORY surpasses any book of the kind yet published.

1. The work is marked by ***uncommon clearness of statement,*** and the most important facts in our history are presented in few words and small space, and in the attractive form of an easy-flowing narrative.

2. The narrative is divided into ***Six Distinct Periods,*** namely: *Discoveries, Settlements, Colonies, The Revolution, The Nation,* and *The Civil War and its Consequences.*

3. The work is ***arranged in Short Sentences,*** so that the substance of each may be easily comprehended.

4. The ***Most Important Events*** are indicated in the text by ***heavy-faced letter.*** All *proper names are printed in italic letter.*

5. ***Full Questions*** are framed for every verse.

6. ***A Pronouncing Vocabulary*** is furnished in foot-notes wherever required.

7. ***A Brief Synopsis*** of topics is given at the close of *each section.*

8. ***An Outline History*** of *important events* is given at the close of *every chapter.*

9. The work is ***Profusely Illustrated*** by maps, charts and plans explanatory of the text, and by carefully-drawn pictures of objects and events.

10. The ***Colonial Seals*** are believed to be the *only strictly accurate ones published,* and have been engraved especially for this book.

11. A few pages devoted to ***Biographical Notes, Facts to be specially remembered,*** and a ***Topical Review*** constitute a valuable feature of the work.